

# Evolution of networks with multiple junctions

Carlo Mantegazza<sup>\*</sup>   Matteo Novaga<sup>†</sup>   Alessandra Pluda<sup>‡</sup>   Felix Schulze<sup>§</sup>

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## Abstract

We consider the motion by curvature of a network of curves in the plane and we discuss existence, uniqueness, singularity formation and asymptotic behavior of the flow.

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<sup>\*</sup>Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, Via Cintia, Monte S. Angelo 80126 Napoli, Italy

<sup>†</sup>Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy

<sup>‡</sup>Fakultät für Mathematik, Universität Regensburg, Universitätsstraße 31, 93053 Regensburg, Germany

<sup>§</sup>Department of Mathematics, University College London, Gower Street, London, UK

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## 1 Introduction

The aim of this work is to give an overview of the state-of-the-art of the motion by curvature of a network of non-intersecting curves in the plane. This problem, which was proposed by Mullins [11] and discussed in [11, 13, 14, 36, 50], attracted the interest of several authors in recent years, see [10, 12, 17, 17, 27, 40, 47, 49, 59, 60, 62–64, 70, 72–75, 82].

One strong motivation to study this flow is the analysis of models of two-dimensional multiphase systems, where the problem of the dynamics of the interfaces between different phases arises naturally. As an example, the model where the energy of a configuration is simply the total length of the interfaces has proven to be useful in the analysis of the growth of grain boundaries in a polycrystalline material, see [11, 36, 50] and <http://mimp.materials.cmu.edu>.

A second motivation is more theoretical: the evolution of such a network of curves is the simplest example of motion by mean curvature of a set which is *essentially* singular. In the literature there are indeed various generalized definitions of flow by mean curvature for non-regular sets (see [2, 13, 21, 28, 45, 79], for instance). Anyway, despite the mean curvature flow of a smooth submanifold being deeply understood, the evolution of generalized submanifolds, possibly singular, for instance *varifolds*, has not been studied in great detail after the seminal work by K. Brakke [13], where the existence of a global (very) weak solution is proved in the *geometric measure theory* context, called “Brakke flow”. In this direction, we also mention the works by T. Ilmanen [44], K. Kasai and Y. Tonegawa [48] and S. Esedoglu and F. Otto [27] (see also [15, 22] for an approach based on the implicit variational scheme introduced in [2, 55]). In particular, in [49], L. Kim and Y. Tonegawa obtain a global existence theorem for the evolution of grain boundaries in  $\mathbb{R}^n$  (which reduces to the evolution of networks when  $n = 2$ ), showing a regularity result when the “density” is less than two. They also show that there exists a finite family of open sets which move continuously with respect to the Lebesgue measure, and whose boundaries coincide with the space-time support of the mean curvature flow. For a global existence result in any codimension and with special regularity properties, adapting the elliptic regularization scheme of Ilmanen, see the work of the last author and B. White [76]. In [82] Y. Tonegawa and N. Wickramasekera adapt the parabolic blow-up method to study the singularities of a Brakke flow of networks and obtain an estimate on the Hausdorff dimension of the singular times.

The definition of the flow is the first problem one has to face, due to the contrast between the intrinsic singular nature of a network and the reasonable desire to have something as “smooth” as possible. Consider for instance the network described by two curves crossing each other, forming a 4-point. There are actually several possible candidates for the flow, even excluding a priori “fattening” phenomena (which can happen for instance in the “level set” approach of L. C. Evans and J. Spruck [28] or, alternatively, Y. G. Chen, Y. Giga and S. Goto [18]). One cannot easily decide how the angles must behave, moreover, it could also be allowed for the four concurrent curves to separate in two pairs of curves moving independently of each other and/or taking into account the possible “birth” of new multi-points from such a single one (all these choices are actually possible with Brakke’s definition).

Actually, one would like that a good/robust definition of curvature flow of a generic initial network should give uniqueness of the motion (at least for “generic” initial data) and forces the network, by a instantaneous regularization, to have only triple junctions, with the possible exception of some discrete

set of times, with the three angles between the concurring curves of 120 degrees. This last property (experimentally observed) is usually called *Herring condition*.

This is actually suggested by the variational nature of the problem, since the flow can be considered as the “gradient flow” in the space of networks of the *Length* functional, which is the sum of the lengths of all the curves of the network (see [13]). It must anyway be said that such a space does not share a natural linear structure and such a “gradient”, in general at the multiple junctions, is not actually a well defined “velocity” vector driving the motion, it only gives that every point of a network different from its multi-points must move with a velocity whose normal component is the curvature vector of the curve it belongs, in order to decrease the *Energy* (that is, the total length here) of the network “most efficiently” (see [13]). From the energetic point of view it is then natural to expect that configurations with multi-points of order greater than three or 3-points with angles different from 120 degrees, being *unstable* for the length functional, should be present only in the initial network or that they should appear only at some discrete set of times, during the flow. This property is also suggested by numerical simulations and physical experiments, see [11, 14, 36, 50] and the *grain growth* movies at <http://facstaff.susqu.edu/brakke>.

One may actually hope that some sort of parabolic regularization could play a role. For instance, if a multi-point has only two concurrent curves, it can be easily shown (see [4, 6, 8, 35]) that the two curves become instantaneously a single smooth curve moving by curvature.

Actually, it is always possible to find a Brakke flow sharing such property at almost every time (see [13]), by the variational spirit of the definition. However, as uniqueness does not hold in this class, there are also Brakke flows starting from the same initial network which keep their multi-points, or loose the connectedness of the network: for instance, a 4-point can “open” as in the right side of Figure 29, or separate in two no more concurring curves, or it can “persist” to be a 4-point where the two “crossing” curves move independently. Moreover, Brakke’s definition is apparently too “weak” (being possibly too general), if one is interested in a detailed description of the flow (see anyway the recent improvements by K. Kasai and Y. Tonegawa [48]).

Another possibility could be to employ a definition of evolution based on *minimizing movements* (see [2] and [55]), that is, a time-discrete variational scheme in which, since the functional to be minimized is a perturbation of the length functional, the minima (at every step) automatically satisfy the Herring condition. Then a conditional compactness theorem would give a limit flow sharing such a property at almost every time, see the recent work [54] by T. Laux and F. Otto.

Due to the importance in this problem, we call networks having only multi-points with three concurrent curves (3-points) forming angles of 120 degrees *regular*.

Following the “energetic” and experimental motivations we thus simply *impose* such a *regularity* condition in the definition of a curvature flow for every positive time (at the initial time it could fail). If the initial network is regular and smooth enough, we will see that this definition leads to an almost satisfactory (in a way “classical”) short time existence theory of a flow by curvature. Considering instead an initial non-regular network, various complications arise related to the presence of multi-points or of 3-points not satisfying the Herring condition.

Anyway, even starting with an initial regular network, we cannot avoid to deal also with non-regular networks when we analyze the global behavior of the flow. Indeed, during the flow some of the triple junctions could “collide” along a “collapsing” curve of the network, when the length of the latter goes to zero (hence, modifying the topological structure of the network). In this case one possibly has to “restart” the evolution with a different set of curves, describing a non-regular network, typically with multi-points of order higher than three (consider, for instance, two 3-points collapsing along a single curve connecting them) or even with “bad” 3-points (think of three 3-points collapsing together along three curves connecting them).

A suitable short time existence (hence, restarting) result has been worked out in [47] by T. Ilmanen, A. Neves and the forth author, where it is shown that starting from any non-regular network (with a natural technical hypothesis), there exists a flow of networks by curvature, which is immediately regular and smooth for every positive time, see Theorem 11.1.

The existence problem of a curvature flow for a regular network with only one 3-point (called a *triod*) was first considered by L. Bronsard and F. Reitich in [14], where they proved the local existence of the flow and by D. Kinderlehrer and C. Liu in [50], who showed the global existence and convergence of a smooth solution if the initial regular triod is sufficiently close to a minimal (Steiner) configuration. An analogous result for networks with Neumann boundary conditions was established by H. Garcke,

Y. Kohsaka and D. Sevcovič in [34].

We mention that the problem in higher dimensions, which is sometimes called in codimension one, *evolution of grain boundaries*, is still widely open. Besides the above mentioned papers [49, 76], where a global weak solution in the Brakke sense is constructed, the short time existence of a smooth and regular solution in three dimensions has been established in [24] and in all dimensions and codimensions in [76], under the assumption that the evolving submanifold has only singular lines (where three surfaces meet at 120 degrees) and no higher order points (where six or more surfaces meet). In these cases, the analysis of singularities and the subsequent possible “restarting” procedure are completely open problems. We also mention the works [29, 30] where a graph evolving by mean curvature and meeting a horizontal hyperplane with a fixed angle of 60 degrees is studied. Then, by considering the union of such graph with its reflection through the hyperplane, one gets an evolving symmetric lens-shaped domain. We remark that in this particular case the analysis is simpler since the maximum principle can be applied.

In higher codimension, an interesting problem is the curvature flow of a network in the Euclidean  $n$ -dimensional space, which was first considered in [76, Section 6], obtaining a short time existence result analogous to Theorem 11.1, but only with initial triple junctions but no higher order junctions.

In this paper, after introducing regular networks, their flow by curvature and having described their basic properties (Sections 2 and 3), we extend in Section 4 a revisited version (see [63]) of the above short time existence theorem of L. Bronsard and F. Reitich to general regular initial networks satisfying some compatibility conditions at the triple junctions (Theorems 4.8 and 4.18). This will be the basis to obtain a short time existence result of the flow, Theorem 6.8, for every initial regular network of class  $C^2$ , in Section 6. To achieve this, we recall in Section 5 the basic estimates (proved in [63]) holding along the evolution of a regular network, which will be used throughout all the paper.

The first consequence of such estimates is that if the lengths of the curves stay away from zero, as  $t$  goes to the maximal time  $T$  of smooth existence of the flow, the maximum of the modulus of the curvature has to diverge (Corollary 5.14).

The problem of the uniqueness of the flow (which is also discussed in Sections 4 and 6) is actually quite delicate and does not have a clear answer at the moment. Even for an initial smooth regular network, we are able to show uniqueness only in the special class of smooth flows, while one should expect uniqueness in the natural class of curvature flows which are simply  $C^2$  in space and  $C^1$  in time (as it happens for the motion by curvature of a closed curve). The difficulty in getting such conclusion is due to the lack of a direct application of the *maximum principle*, which is the main tool to get estimates on the geometric quantities during the flow, due to the presence of the 3-points which behave as “boundary” points (whereas from a “distributional point of view” they more behave like “interior” points).

After discussing the existence and uniqueness of a curvature flow on a maximal time interval, we try to generalize the analysis of the motion by curvature of closed curves in the Euclidean plane, employing a mix of PDE’s, variational and differential geometry techniques. To this aim, in Section 7 we recall Huisken’s monotonicity formula for mean curvature flow, which holds also for the evolution of a network and we introduce the rescaling procedures used to get blow-up limits at the maximal time of smooth existence (discussed in Section 8), in order to describe the singularities of the flow (and possibly exclude them, under some hypotheses).

One can reasonably expect, for instance, that an embedded regular network does not actually develop singularities during the flow if its “topological structure” does not change because of the “collision” of two or more 3-points. Our analysis in Sections 8 and 9 will show that if no multiplicities larger than one occur, this expectation is indeed true (in some special cases such “bad” multiplicities can be ruled out, see Section 14). Essentially, one needs to classify the possible blow-up limits at the singular time, in order to actually exclude them by means of geometric arguments. Some key references for this method in the case of a single smooth curve are [3, 38, 42, 43], the most relevant difference in dealing with networks is the difficulty in using the maximum principle, which is the main tool to get point-wise estimates on the geometric quantities during the flow. For this reason, some important estimates which are straightforward in the smooth case are here more complicated to prove (and sometimes we do not know if they actually hold) and one has to resort to integral estimates. These latter are similar to the ones in [3, 6, 8, 41], for instance, but require some extra work in order to deal with the triple junctions.

Under the assumption that the lengths of the curves are bounded away from zero and no multiplicities larger than one occur, the only possible blow-up limits are given by a straight line, a halfline

with multiplicity one, or a flat unbounded regular triod (called “standard triod”) composed by three halflines for the origin of  $\mathbb{R}^2$  forming angles of 120 degrees, see Proposition 8.28 and Section 10. Then, a local regularity theorem for the flow (shown in [47]), together with such a classification, excludes the presence of singularities. Such a theorem, which is in the spirit of White’s local regularity theorem for mean curvature flow [84], is presented in detail in Section 9.

Section 10 is devoted to analyze the behavior of the flow of a general network approaching a singular time: as we said, some lengths of the curves composing the network may not be bounded away from zero and we discuss the properties of the possible limit networks. The case in which the curvature is not bounded is clearly the most complicated (see Section 10.3), we are anyway able to show that if only two triple junctions collide along a collapsing curve, forming a 4-point, the curvature remains bounded. The situation in which an entire region collapses is instead much more difficult to treat. Assuming the uniqueness of the blow-up limit along any sequence of rescalings, we can prove that, as  $t \rightarrow T$ , the network  $S_t$  converges to some degenerate (see Definition 8.1) regular network, whose “non-collapsed” part  $S_T$  is a  $C^1$ , possibly non-regular, network which is smooth outside its multi-points and the curvature is of order  $o(1/r)$ , where  $r$  is the distance from its non-regular multi-points.

With such an analysis, in Section 11 it is explained how (under the above mentioned assumptions, which are actually still open conjectures) one can restart the flow after a singularity by means of an existence theorem for the flow, starting from a non-regular network (Theorem 11.1). We include an outline of the argument in [47], showing that for a network with multi-points of order greater than 3 and/or non-regular 3-points, there exists a smooth curvature flow for short time only having regular 3-points at every positive time. The idea is to locally desingularise the multi-points and the non-regular 3-points via regular self-similarly expanding solutions. The argument hinges on a new monotonicity formula, which shows that such expanding solutions are dynamically stable, using the fact that the evolution of curves and networks in the plane are special cases of Lagrangian mean curvature flow (these ideas have already been exploited by A. Neves in the papers [66–68]).

In Section 12 we discuss the restarting of the curvature flow of a network after a singularity by means of Theorem 11.1. We analyze the preserved geometric quantities and the possible changes in the topology of a network in passing through a singularity. This is applied in Section 13 to study the long time behavior of the flow and its possible asymptotic convergence (up to a subsequence of times) to a critical point of the length functional (a Steiner network).

The restarting procedure allows us to define an “extended” curvature flow with singularities at an increasing sequence of times. An important open question is whether the maximal time interval of existence of such flow is finite or not, where the main problem is the possible “accumulation” of the singular times (if they are not finite, which actually we do not know). Clearly, in case such an “extended” flow can be defined for every time (as the Brakke flow obtained by L. Kim and Y. Tonegawa in [49]), like for a flow without singularities after some time, we ask ourselves if the network converges, as  $t \rightarrow +\infty$ , to a stationary network for the length functional.

In Section 14 we discuss a scaling invariant, geometric quantity associated to a network, first proposed in [39] (see also [43]), later extended in [63] to the case of a triod and in [12] to lens-shaped networks, consisting in a sort of “embeddedness measure” which is positive when no self-intersections are present. By a monotonicity argument, we show that this quantity is bounded below along the flow, under the assumption that the number of 3-points of the network is not greater than two. As a consequence, in this case we have that every possible  $C_{\text{loc}}^1$ -limit of rescalings of networks of the flow is an embedded network with multiplicity one. We underline that it is not clear to us how to obtain a similar bound/conclusion for a general network (with several triple junctions), since the analogous quantity, if there are more than two 3-points, (apparently) does not share the above monotonicity property.

In Section 15, under the assumption that there is only one 3-point, we can describe the whole evolution of a network in detail: in the case of a triod we show its convergence to a Steiner network if the lengths of the three curves stay bounded away from zero and in the case of a spoon we describe the formation of the possible singularities.

We conclude the paper with Section 16, dedicated to the main open problems and, by courtesy of Tom Ilmanen, with an appendix with pictures and computations of several examples of regular shrinkers, due to him and Jörg Hättenschweiler.

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## 2 Notation and definitions

Given a  $C^1$  curve  $\sigma : [0, 1] \rightarrow \mathbb{R}^2$  we say that it is *regular* if  $\sigma_x = \frac{d\sigma}{dx}$  is never zero. It is then well defined its unit tangent vector  $\tau = \sigma_x / |\sigma_x|$ . We define its unit normal vector as  $\nu = R\tau = R\sigma_x / |\sigma_x|$ , where  $R : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is the counterclockwise rotation centered in the origin of  $\mathbb{R}^2$  of angle  $\pi/2$ .

If the curve  $\sigma$  is  $C^2$  and regular, its *curvature vector* is well defined as  $\underline{k} = \tau_x / |\sigma_x| = \frac{d\tau}{dx} / |\sigma_x|$ .

The arclength parameter of a curve  $\sigma$  is given by

$$s = s(x) = \int_0^x |\sigma_x(\xi)| d\xi.$$

Notice that  $\partial_s = |\sigma_x|^{-1} \partial_x$ , then  $\tau = \partial_s \sigma$  and  $\underline{k} = \partial_s \tau$ , hence, the curvature of  $\sigma$  is given by  $k = \langle \underline{k} | \nu \rangle$ , as  $\underline{k} = k\nu$ .

**Definition 2.1.** Let  $\Omega$  be a smooth, convex, open set in  $\mathbb{R}^2$ . A *network*  $\mathbb{S} = \bigcup_{i=1}^n \sigma^i([0, 1])$  in  $\Omega$  is a connected set in the plane described by a finite family of  $C^1$ , regular curves  $\sigma^i : [0, 1] \rightarrow \Omega$  such that

1. the “interior” of every curve  $\sigma^i$ , that is  $\sigma^i(0, 1)$ , is embedded (hence, it has no self-intersections); a curve can self-intersect itself only possibly “closing” at its end-points;
2. two different curves can intersect each other only at their end-points;
3. if a curve of the network touches the boundary of  $\Omega$  at a point  $P$ , no other end-point of a curve can coincide with that point.

We call *multi-points of the network* the vertices  $O^1, O^2, \dots, O^m \in \Omega$ , seeing  $\mathbb{S}$  as a planar graph, where the order is greater than one.

We call *end-points of the network*, the vertices  $P^1, P^2, \dots, P^l \in \overline{\Omega}$  of  $\mathbb{S}$  (on the boundary or not) with order one.

We say that a network is of class  $C^k$  or  $C^\infty$  if all the  $n$  curves are respectively  $C^k$  or  $C^\infty$ .

*Remark 2.2.*

- In the special case that a network consists of a single closed embedded curve, its motion by curvature was widely studied by many authors (see [6–8, 31–33, 35]), the curves evolves smoothly, becomes convex and shrinks down to a point in finite time, getting rounder and rounder. Moreover, also the case that the curve has an angle or a cusps (where the cusp is the most “delicate” situation) can be dealt with by means of the works of Angenent [6–8], actually the curve becomes immediately smooth, flowing by curvature.
- When two curves concur at a 2-point of the network forming an angle (or a cusp, if they have the same tangent), such situation that can be analyzed as we said above, by considering them as a single curve with a “singular” point that vanishes immediately under the flow.
- If a network is composed by a single embedded curve with fixed end-points, its evolution by curvature is discussed in [43, 80, 81]. The curve converges to the straight segment connecting the two fixed end-points in infinite time.

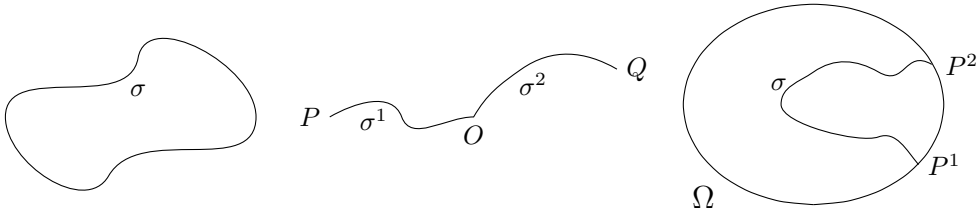


Figure 1: Three special situations: a single closed curve, two curves forming an angle at their junction and a single curve with two end-points on the boundary of  $\Omega$ .

- Condition 3 about the curves at the boundary is to keep things simple and implies that the multi-points can be present only inside  $\Omega$ , not on the boundary, while the end-points can be both inside or on  $\partial\Omega$ .

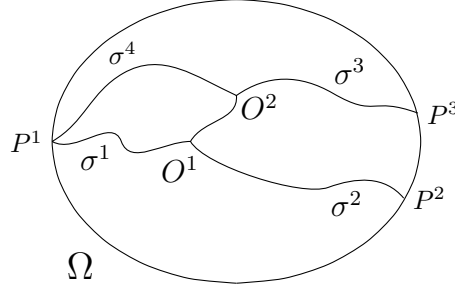


Figure 2: An example of “violation” of Condition 3 in the definition of network.

The curves  $\sigma^i$  have clearly nonzero finite lengths  $L^i = \int_0^1 |\sigma_x^i(\xi)| d\xi$  and we will denote with  $L = L^1 + \dots + L^n$  the *global length* of the network.

**Definition 2.3.** An *open network*  $\mathbb{S} = \bigcup_{i=1}^n \sigma^i(I)$  in  $\mathbb{R}^2$  is a connected set in the plane described by a finite family of  $C^1$ , regular curves  $\sigma^i : I \rightarrow \mathbb{R}^2$ , where  $I$  can be the interval  $[0, 1]$  or  $[0, 1)$ , such that

1. every “open” curve  $\sigma^i : [0, 1) \rightarrow \mathbb{R}^2$  is  $C^1$ –asymptotic to a halfline in  $\mathbb{R}^2$  as  $x \rightarrow 1$ ;
2. the “interior” of every curve  $\sigma^i$  is embedded (hence, it has no self–intersections), only a curve of the kind  $\sigma^i : [0, 1] \rightarrow \mathbb{R}^2$  can self–intersect itself and only “closing” at its end–points;
3. two different curves can intersect each other only at their end–points;
4. every end–point of a curve belongs to some multi–point of the network with order at least two, considering  $\mathbb{S}$  as a planar graph;

As before, we say that an open network is of class  $C^k$  or  $C^\infty$  if all the  $n$  curves are respectively  $C^k$  or  $C^\infty$ .

*Remark 2.4.* Since we called these unbounded networks “open”, we will adopt the word “closed” for the previous networks in Definition 2.1 which are bounded and possibly have some end–points.

Given a network composed by  $n$  curves with  $l$  end–points  $P^1, P^2, \dots, P^l \in \overline{\Omega}$  (if present) and  $m$  multi–points  $O^1, O^2, \dots, O^m \in \Omega$ , we will denote with  $\sigma^{pi}$  the curves of this network concurring at the multi–point  $O^p$ , with the index  $i$  varying from one to the order of the multi–point  $O^p$  (this is clearly redundant as some curves coincides, but useful for the notation). In the case of a network of  $n$  curves with only  $m$  3–points, it is then composed by the family (with possible repetitions) of curves  $\sigma^{pi}$ , with  $p \in \{1, 2, \dots, m\}$  and  $i \in \{1, 2, 3\}$ .

Our goal will be to analyze the curvature flow of a network assuming either it is open or that all its end–points (if present) have to coincide with some points  $P^1, P^2, \dots, P^l$  on the boundary of  $\Omega$  (as we said, by Condition 3 in Definition 2.1, at most one curve of the network can arrive at any point  $P^r$ ). We will discuss existence, uniqueness, regularity and asymptotic behavior of the evolution by curvature of such a network.

In the “closed” case we will ask that the end–points  $P^r \in \partial\Omega$  stay “fixed” (*Dirichlet boundary conditions*) during the evolution. An analogous problem is to let such end–points “free” to move on the boundary of  $\Omega$  but asking that the curves intersect orthogonally  $\partial\Omega$  (*Neumann boundary conditions*).

We will define now a special class of networks that will play a key role in the analysis.

**Definition 2.5.** We call a network (open or not) *regular* if all its multi–points  $O^1, O^2, \dots, O^m \in \Omega$  have order three and at each of them the three concurring curves  $\{\sigma^{pi}\}_{i=1,2,3}$  meet in such a way that the external unit tangents  $\tau^{pi}$  satisfy  $\tau^{p1} + \tau^{p2} + \tau^{p3} = 0$ , which means that the three curves form three angles of 120 degrees at  $O^p$  (*Herring condition*).

We call a network *non-regular* if some multi-point has order different from three or if it has order three but the external unit tangents of the three concurring curves  $\{\sigma^{p^i}\}_{i=1,2,3}$  do not satisfy  $\tau^{p^1} + \tau^{p^2} + \tau^{p^3} = 0$ . We will call such a point a *non-regular multi-point*.

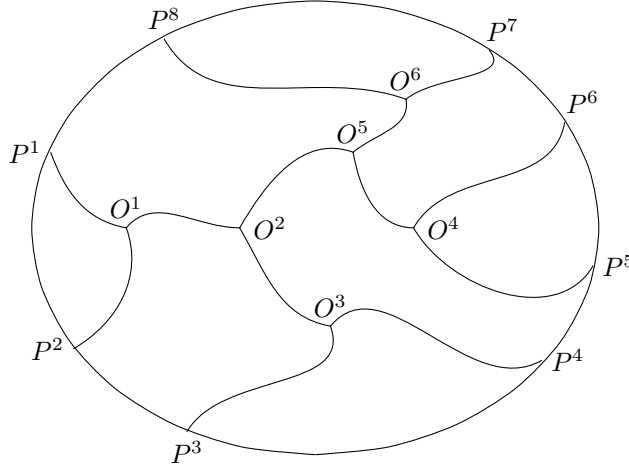


Figure 3: A regular network.

We are now ready to define the evolution by curvature of a  $C^2$  regular network, which, in the “closed case”, is the *geometric gradient flow* of the *length functional*, that is, the sum of the lengths of all the curves of the network. Roughly speaking, a (solution of the) *flow by curvature* of a network is a smooth family of embedded, planar networks, such that the normal component of the velocity under the evolution, at every point of every curve of the evolving network, is given by the curvature vector of the curve at the point.

Given a time-dependent family of regular  $C^2$  networks of curves  $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i([0, 1], t)$ , we let  $\tau^i = \tau^i(x, t)$  the unit tangent vector to the curve  $\gamma^i$ ,  $\nu^i = \nu^i(x, t) = R\tau^i(x, t)$  the unit normal vector and  $\underline{k}^i = \underline{k}^i(x, t) = k^i(x, t)\nu^i(x, t)$  its curvature vector, as previously defined.

Here and in all the sequel, we will denote with  $\partial_x f$ ,  $\partial_s f$  and  $\partial_t f$  the derivatives of a function  $f$  along a curve  $\gamma^i$  with respect to the  $x$  variable, the arclength parameter  $s$  on such curve, defined by  $s(x, t) = \int_0^x |\gamma_x^i(\xi, t)| d\xi$ , and the time;  $\partial_x^n f$ ,  $\partial_s^n f$ ,  $\partial_t^n f$  are the higher order partial derivatives which often we will also write as  $f_x, f_{xx}, \dots, f_s, f_{ss}, \dots$  and  $f_t, f_{tt}, \dots$ .

**Definition 2.6.** We say that a family of homeomorphic, regular networks  $\mathbb{S}_t$ , each one composed by  $n$  curves  $\gamma^i(\cdot, t) : I_i \rightarrow \overline{\Omega}$  (where  $I_i$  is the interval  $[0, 1]$  or  $[0, 1)$  in case of an open network), in a smooth convex, open set  $\Omega \subset \mathbb{R}^2$ , *moves by curvature* in the time interval  $(0, T)$  if the functions  $\gamma^i : I_i \times (0, T) \rightarrow \overline{\Omega}$  are of class  $C^2$  in space and  $C^1$  in time at least and satisfy

$$\begin{aligned} \gamma_t^i(x, t) &= k^i(x, t)\nu^i(x, t) + \lambda^i(x, t)\tau^i(x, t) \\ &= \frac{\langle \gamma_{xx}^i(x, t) | \nu^i(x, t) \rangle}{|\gamma_x^i(x, t)|^2} \nu^i(x, t) + \lambda^i(x, t)\tau^i(x, t) \end{aligned} \quad (2.1)$$

for some continuous functions  $\lambda^i$ , for every  $x \in I_i$ ,  $t \in (0, T)$ ,  $i \in \{1, 2, \dots, n\}$ .

Another equivalent way to state this evolution equation is clearly

$$\gamma_t^i(x, t)^\perp = k^i(x, t)\nu^i(x, t) = \underline{k}^i(x, t) = \frac{\langle \gamma_{xx}^i(x, t) | \nu^i(x, t) \rangle}{|\gamma_x^i(x, t)|^2} \nu^i(x, t).$$

We will call  $\underline{v}^i = \gamma_t^i = k^i\nu^i + \lambda^i\tau^i$  and  $\underline{\lambda}^i = \lambda^i\tau^i$  respectively the *velocity* and the *tangential velocity* of the curve  $\gamma^i$ , notice that the *normal velocity* is given by the curvature vector of the curve  $\gamma^i$  at every point. It is easy to see that  $\underline{v}^i = \underline{k}^i + \underline{\lambda}^i$  and  $|\underline{v}^i|^2 = |\underline{k}^i|^2 + |\underline{\lambda}^i|^2 = (k^i)^2 + (\lambda^i)^2$ .



We underline that, in general, if there is no need to make explicit the curves composing a network, we simply write  $\tau, \nu, \underline{v}, \underline{k}, \underline{\lambda}, k, \lambda$  for the previous quantities, omitting the indices. Moreover, we will adopt the following convention for integrals,

$$\int_{\mathbb{S}_t} f(t, \gamma, \tau, \nu, k, k_s, \dots, \lambda, \lambda_s \dots) ds = \sum_{i=1}^n \int_0^1 f(t, \gamma^i, \tau^i, \nu^i, k^i, k_s^i, \dots, \lambda^i, \lambda_s^i \dots) |\gamma_x^i| dx$$

as the arclength measure is given by  $ds = |\gamma_x^i| dx$  on every curve  $\gamma^i$ .

Sometimes we will use also the following notation for the evolution of a network in  $\Omega \subset \mathbb{R}^2$ : we let  $\mathbb{S} \subset \mathbb{R}^2$  a “referring” network homeomorphic to the all  $\mathbb{S}_t$  and we consider a map  $F : \mathbb{S} \times (0, T) \rightarrow \mathbb{R}^2$  given by the “union” of the maps  $\gamma^i : I_i \times (0, T) \rightarrow \overline{\Omega}$  describing the curvature flow of the network in the time interval  $(0, T)$ , that is,  $\mathbb{S}_t = F(\mathbb{S}, t)$

*Remark 2.7.* We spend some words on the above definition and on the evolution equation (2.1) which is not the usual way to describe the motion by curvature of a smooth curve, that is,

$$\gamma_t^i = \underline{k}^i = k^i \nu^i = \frac{\langle \gamma_{xx}^i | \nu^i \rangle}{|\gamma_x^i|^2} \nu^i. \quad (2.2)$$

Both motions are driven by a system of quasilinear partial differential equations, in our definition “admitting a correction” by a tangential term. Indeed, the two velocities differ only for a tangential component  $\underline{\lambda}^i = \lambda^i \tau^i$ . In the curvature evolution of a smooth curve it is well known that any tangential contribution to the velocity actually affects only the “inner motion” of the “single points” (Lagrangian point of view), but it does not affect the motion of a curve as a whole subset of  $\mathbb{R}^2$ , forgetting its parametrization (Eulerian point of view). Indeed, it can be shown that a flow of a closed curve satisfying equation (2.1) can be globally reparametrized (dynamically in time) in order it satisfies equation (2.2). However, in our situation such a global reparametrization is not possible due to the presence of the 3-points. It is necessary to consider such extra tangential terms in order to allow the motion of the 3-points also. Indeed, if the velocity would be in normal direction at every point of the three curves concurring at a 3-point, this latter should move in a direction which is normal to all of them, then the only possibility would be that it does not move at all (see also the discussions and examples in [13, 14, 50]).

*Remark 2.8.* A very special case of an evolving curve  $\gamma^i$  satisfying equation (2.1) is a solution of the following system of quasilinear partial differential equations,

$$\gamma_t^i = \frac{\gamma_{xx}^i}{|\gamma_x^i|^2}.$$

In this case, it follows that

$$\begin{aligned} \underline{v}^i &= \underline{v}^i(x, t) = \frac{\gamma_{xx}^i}{|\gamma_x^i|^2} && \text{velocity of the point } \gamma^i(x, t), \\ \lambda^i &= \lambda^i(x, t) = \frac{\langle \gamma_{xx}^i | \tau^i \rangle}{|\gamma_x^i|^2} = \frac{\langle \gamma_{xx}^i | \gamma_x^i \rangle}{|\gamma_x^i|^3} = -\partial_x \frac{1}{|\gamma_x^i|} && \text{tangential velocity of the point } \gamma^i(x, t), \\ k^i &= k^i(x, t) = \frac{\langle \gamma_{xx}^i | \nu^i \rangle}{|\gamma_x^i|^2} = \langle \partial_s \tau^i | \nu^i \rangle = -\langle \partial_s \nu^i | \tau^i \rangle && \text{curvature at the point } \gamma^i(x, t). \end{aligned}$$

**Definition 2.9.** A curvature flow  $\gamma^i$  for the initial, regular  $C^2$  network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$  which satisfies  $\gamma_t^i = \frac{\gamma_{xx}^i}{|\gamma_x^i|^2}$  for every  $t > 0$  will be called a *special curvature flow* of  $\mathbb{S}_0$ .

**Definition 2.10.** Given an initial, regular,  $C^2$  network  $\mathbb{S}_0$ , composed by  $n$  curves  $\sigma^i : [0, 1] \rightarrow \overline{\Omega}$ , with  $m$  3-points  $O^1, O^2, \dots, O^m \in \Omega$  and (if present)  $l$  end-points  $P^1, P^2, \dots, P^l \in \partial\Omega$  in a smooth convex, open set  $\Omega \subset \mathbb{R}^2$ , we say that a family of homeomorphic networks  $\mathbb{S}_t$ , described by the family of time-dependent curves  $\gamma^i(\cdot, t)$ , is a solution of the motion by curvature problem with fixed end-points in the time interval  $[0, T)$  if the functions  $\gamma^i : [0, 1] \times [0, T) \rightarrow \overline{\Omega}$  are continuous, there holds  $\gamma^i(x, 0) = \sigma^i(x)$  for every  $x \in [0, 1]$  and  $i \in \{1, 2, \dots, n\}$  (initial data), they are at least  $C^2$  in space and  $C^1$  in time in  $[0, 1] \times (0, T)$  and satisfy the following system of conditions for every  $x \in [0, 1]$ ,  $t \in (0, T)$ ,  $i \in \{1, 2, \dots, n\}$ ,

$$\left\{ \begin{array}{ll} \gamma_x^i(x, t) \neq 0 & \text{regularity} \\ \gamma^r(1, t) = P^r & \text{with } 0 \leq r \leq l \quad \text{end-points condition} \\ \sum_{j=1}^3 \tau^{pj}(O^p, t) = 0 & \text{at every 3-point } O^p \quad \text{angles of 120 degrees} \\ \gamma_t^i = k^i \nu^i + \lambda^i \tau^i & \text{for some continuous functions } \lambda^i \quad \text{motion by curvature} \end{array} \right. \quad (2.3)$$

where we assumed conventionally (possibly reordering the family of curves and “inverting” their parametrization) that the end-point  $P^r$  of the network is given by  $\gamma^r(1, t)$  (by Condition 3 in Definition 2.1 this can be always done).

Moreover, in the third equation we abused a little the notation, denoting with  $\tau^{pj}(O^p, t)$  the respective exterior unit tangent vectors at  $O^p$  of the three curves  $\gamma^{pj}(\cdot, t)$  in the family  $\{\gamma^i(\cdot, t)\}$  concurring at the 3-point  $O^p$ .

We also state the same problem for regular, open networks.

**Definition 2.11.** Given an initial, regular,  $C^2$  open network  $\mathbb{S}_0$ , composed by  $n$  curves  $\sigma^i : I_i \rightarrow \mathbb{R}^2$ , we say that a family of homeomorphic open networks  $\mathbb{S}_t$  with the same structure as  $\mathbb{S}_0$  (in particular, same asymptotic halflines at infinity), described by the family of time-dependent curves  $\gamma^i(\cdot, t)$ , is a solution of the motion by curvature problem in the time interval  $[0, T)$  if the functions  $\gamma^i : I_i \times [0, T) \rightarrow \mathbb{R}^2$  are continuous, there holds  $\gamma^i(x, 0) = \sigma^i(x)$  for every  $x \in I_i$  and  $i \in \{1, 2, \dots, n\}$  (initial data), they are of class at least  $C^2$  in space and  $C^1$  in time in  $I_i \times (0, T)$  (here  $I_i$  denotes the interval  $[0, 1]$  or  $[0, 1)$  depending whether the curve is unbounded or not) and satisfy the following system for every  $x \in I_i$ ,  $t \in (0, T)$ ,  $i \in \{1, 2, \dots, n\}$ ,

$$\begin{cases} \gamma_x^i(x, t) \neq 0 & \text{regularity} \\ \sum_{j=1}^3 \tau^{pj}(O^p, t) = 0 & \text{at every 3-point } O^p \quad \text{angles of 120 degrees} \\ \gamma_t^i = k^i \nu^i + \lambda^i \tau^i & \text{for some continuous functions } \lambda^i \quad \text{motion by curvature} \end{cases} \quad (2.4)$$

where, in the second equation, we used the same notation as in Definition 2.10.

*Remark 2.12.* In Definitions 2.10 and 2.11 the evolution equation (2.1) must be satisfied till the borders of the intervals  $[0, 1]$  and  $[0, 1)$ , that is, at the 3-points and the at end-points, for every positive time. This is not the usual way to state boundary conditions for parabolic problems (the parabolic nature of this evolution problem is clear by Definition 2.6 – see also Remark 2.7 and it will be even clearer in Section 4), where usually only continuity at the boundary is required. Anyway, as it is common in parabolic problems, at every positive time such boundary conditions are satisfied by any “natural solution”.

This property of regularity at the boundary implies that

$$(k\nu + \lambda\tau)(P^r) = 0, \text{ for every } r \in \{1, 2, \dots, l\}$$

and

$$(k^{pi}\nu^{pi} + \lambda^{pi}\tau^{pi})(O^p) = (k^{pj}\nu^{pj} + \lambda^{pj}\tau^{pj})(O^p), \text{ for every } i, j \in \{1, 2, 3\}, p \in \{1, 2, \dots, m\}$$

(where we abused a little the notation), obtained by simply requiring that the velocity is zero at every end-point and it is the same for any three curves at their concurrency 3-point.

Moreover, notice that in Definitions 2.10 and 2.11 the evolution equation (2.1) must be satisfied only for  $t > 0$ . If we want that the maps  $\gamma^i$  are  $C^2$  in space and  $C^1$  in time till the whole *parabolic boundary* (given by  $[0, 1] \times \{0\} \cup \{0, 1\} \times [0, T)$  in Definition 2.10 and  $[0, 1] \times \{0\} \cup \{0, 1\} \times [0, T)$  or  $[0, 1) \times \{0\} \cup \{0\} \times [0, T)$  in Definition 2.11), the above conditions must be satisfied also by the initial regular network  $\mathbb{S}_0$ , for some functions  $\lambda_0$  extending continuously the functions  $\lambda$  which are defined only for  $t > 0$ .

We concentrated on regular network for the moment since in studying problems (2.3) and (2.4) starting from a non-regular network several difficulties arise, related to the presence of general multi-points: if there are multi-points  $O^p$  of order greater than three, there can be several possible candidates for the flow. Considering, for example, the case of a network composed by two curves crossing each other (presence of 4-point); one cannot easily decide how the angle at the meeting point must behave, indeed one can allow the four concurrent curves to separate in two pairs of curves, moving independently each other and could even be taken into account the creation of new multi-points from a single one. If there are several multi-points during the flow some of them can collapse together and the length of at least one curve of the network can go to zero.

In these cases, one must possibly restart the evolution with a different set of curves, the topology of the

network change dramatically, forcing to change the “structure” of the system of equations governing the evolution.

Anyway a very natural conjecture is that the curvature flow of a general network (under a suitably good definition) should be non-regular only for a discrete set of times. We will get back on this in the following sections.

### 3 Basic computations

We work out now some basic relations and formulas holding for a regular network evolving by curvature, assuming that all the derivatives of the functions  $\gamma^i$  and  $\lambda^i$  that appear exist.

**Lemma 3.1.** *If  $\gamma$  is a curve moving by*

$$\gamma_t = k\nu + \lambda\tau,$$

*then the following commutation rule holds,*

$$\partial_t \partial_s = \partial_s \partial_t + (k^2 - \lambda_s) \partial_s. \quad (3.1)$$

*Proof.* Let  $f : [0, 1] \times [0, T) \rightarrow \mathbb{R}$  be a smooth function, then

$$\begin{aligned} \partial_t \partial_s f - \partial_s \partial_t f &= \frac{f_{tx}}{|\gamma_x|} - \frac{\langle \gamma_x | \gamma_{xt} \rangle f_x}{|\gamma_x|^3} - \frac{f_{tx}}{|\gamma_x|} = -\langle \tau | \partial_s \gamma_t \rangle \partial_s f \\ &= -\langle \tau | \partial_s (\lambda\tau + k\nu) \rangle \partial_s f = (k^2 - \lambda_s) \partial_s f \end{aligned}$$

and the formula is proved.  $\square$

Then we can compute, for an evolving curve as in the previous lemma,

$$\begin{aligned} \partial_t \tau &= \partial_t \partial_s \gamma = \partial_s \partial_t \gamma + (k^2 - \lambda_s) \partial_s \gamma = \partial_s (\lambda\tau + k\nu) + (k^2 - \lambda_s) \tau = (k_s + k\lambda) \nu, \\ \partial_t \nu &= \partial_t (R\tau) = R \partial_t \tau = -(k_s + k\lambda) \tau, \\ \partial_t k &= \partial_t \langle \partial_s \tau | \nu \rangle = \langle \partial_t \partial_s \tau | \nu \rangle + (k^2 - \lambda_s) \langle \partial_s \tau | \nu \rangle \\ &= \partial_s \langle \partial_t \tau | \nu \rangle + k^3 - k\lambda_s = \partial_s (k_s + k\lambda) + k^3 - k\lambda_s \\ &= k_{ss} + k_s \lambda + k^3. \end{aligned} \quad (3.2)$$

Moreover, in the special case that  $\lambda = \frac{\langle \gamma_{xx} | \gamma_x \rangle}{|\gamma_x|^3}$ , when the curve  $\gamma$  evolves according to

$$\gamma_t = \frac{\gamma_{xx}}{|\gamma_x|^2} = k\nu + \lambda\tau,$$

(see Remark 2.8) we can also compute

$$\begin{aligned} \partial_t \lambda &= -\partial_t \partial_x \frac{1}{|\gamma_x|} = \partial_x \frac{\langle \gamma_x | \gamma_{tx} \rangle}{|\gamma_x|^3} = \partial_x \frac{\langle \tau | \partial_s (\lambda\tau + k\nu) \rangle}{|\gamma_x|} = \partial_x \frac{(\lambda_s - k^2)}{|\gamma_x|} \\ &= \partial_s (\lambda_s - k^2) - \lambda (\lambda_s - k^2) = \lambda_{ss} - \lambda \lambda_s - 2k k_s + \lambda k^2. \end{aligned}$$

We consider the curvature flow of a family of regular,  $C^\infty$  networks  $\mathbb{S}_t$ , composed by  $n$  curves  $\gamma^i$  with  $m$  3-points  $O^1, O^2, \dots, O^m$  and  $l$  end-points  $P^1, P^2, \dots, P^l$ .

Then, at every 3-point  $O^p$ , with  $p \in \{1, 2, \dots, m\}$ , differentiating in time the concurrency condition

$$\gamma^{pi}(0, t) = \gamma^{pj}(0, t) \quad \text{for every } i \text{ and } j,$$

where  $\gamma^{pi}$  denotes the  $i$ -th curve concurrent at the 3-point  $O^p$  and we supposed for simplicity that they are parametrized such that they all concur for  $x = 0$  at  $O^p$ , we get

$$\lambda^{pi} \tau^{pi} + k^{pi} \nu^{pi} = \lambda^{pj} \tau^{pj} + k^{pj} \nu^{pj},$$

at every 3-point  $O^p$ , with  $p \in \{1, 2, \dots, m\}$ , for every  $i, j \in \{1, 2, 3\}$ .

Multiplying these vector equalities for  $\tau^{pl}$  and  $\nu^{pl}$  and varying  $i, j, l$ , thanks to the conditions  $\sum_{i=1}^3 \tau^{pi} = \sum_{i=1}^3 \nu^{pi} = 0$ , we get the relations

$$\begin{aligned}\lambda^{pi} &= -\lambda^{p(i+1)}/2 - \sqrt{3}k^{p(i+1)}/2 \\ \lambda^{pi} &= -\lambda^{p(i-1)}/2 + \sqrt{3}k^{p(i-1)}/2 \\ k^{pi} &= -k^{p(i+1)}/2 + \sqrt{3}\lambda^{p(i+1)}/2 \\ k^{pi} &= -k^{p(i-1)}/2 - \sqrt{3}\lambda^{p(i-1)}/2\end{aligned}$$

with the convention that the second superscripts are to be considered “modulus 3”. Solving this system we get

$$\begin{aligned}\lambda^{pi} &= \frac{k^{p(i-1)} - k^{p(i+1)}}{\sqrt{3}} \\ k^{pi} &= \frac{\lambda^{p(i+1)} - \lambda^{p(i-1)}}{\sqrt{3}}\end{aligned}$$

which implies

$$\sum_{i=1}^3 k^{pi} = \sum_{i=1}^3 \lambda^{pi} = 0 \quad (3.3)$$

at any 3-point  $O^p$  of the network  $\mathbb{S}_t$ .

Moreover, considering  $K^p = (k^{p1}, k^{p2}, k^{p3})$  and  $\Lambda^p = (\lambda^{p1}, \lambda^{p2}, \lambda^{p3})$  as vectors in  $\mathbb{R}^3$ , we have seen that  $K^p$  and  $\Lambda^p$  belong to the plane orthogonal to the vector  $(1, 1, 1)$  and

$$K^p = \Lambda^p \wedge (1, 1, 1)/\sqrt{3}, \quad \Lambda^p = -K^p \wedge (1, 1, 1)/\sqrt{3},$$

that is,  $K^p = S\Lambda^p$  and  $\Lambda^p = -SK^p$  where  $S$  is the rotation in  $\mathbb{R}^3$  of an angle of  $\pi/2$  around the axis  $I = \langle(1, 1, 1)\rangle$ . Hence, it also follows that

$$\sum_{i=1}^3 (k^{pi})^2 = \sum_{i=1}^3 (\lambda^{pi})^2 \quad \text{and} \quad \sum_{i=1}^3 k^{pi} \lambda^{pi} = 0.$$

at any 3-point  $O^p$  of the network  $\mathbb{S}_t$ .

Now we differentiate in time the angular condition  $\sum_{i=1}^3 \tau^{pi} = 0$  at every 3-point  $O^p$ , with  $p \in \{1, 2, \dots, m\}$ , with by equation (3.2) we get

$$k_s^{pi} + \lambda^{pi} k^{pi} = k_s^{pj} + \lambda^{pj} k^{pj},$$

for every pair  $i, j$ . In terms of vectors in  $\mathbb{R}^3$ , as before, we can write

$$K_s^p + \Lambda^p K^p = (k_s^{p1} + \lambda^{p1} k^{p1}, k_s^{p2} + \lambda^{p2} k^{p2}, k_s^{p3} + \lambda^{p3} k^{p3}) \in I.$$

Differentiating repeatedly in time all these vector relations we have

$$\begin{aligned}\partial_t^l K^p, \partial_t^l \Lambda^p &\perp I \quad \text{and} \quad \partial_t^l \langle K^p | \Lambda^p \rangle = 0, \\ \partial_t^l \Lambda^p &= -\partial_t^l S K^p = -S \partial_t^l K^p, \\ \partial_t^m (K_s^p + \Lambda^p K^p) &\in I,\end{aligned} \quad (3.4)$$

which, making explicit the indices, give the following equalities at every 3-point  $O^p$ , with  $p \in \{1, 2, \dots, m\}$ ,

$$\begin{aligned}\partial_t^l \sum_{i=1}^3 k^{pi} &= \sum_{i=1}^3 \partial_t^l k^{pi} = \partial_t^l \sum_{i=1}^3 \lambda^{pi} = \sum_{i=1}^3 \partial_t^l \lambda^{pi} = \partial_t \sum_{i=1}^3 k^{pi} \lambda^{pi} = 0, \\ \sum_{i=1}^3 (\partial_t^l k^{pi})^2 &= \sum_{i=1}^3 (\partial_t^l \lambda^{pi})^2 \quad \text{for every } l \in \mathbb{N},\end{aligned}$$

$$\partial_t^m(k_s^{pi} + \lambda^{pi} k^{pi}) = \partial_t^m(k_s^{pj} + \lambda^{pj} k^{pj}) \text{ for every pair } i, j \text{ and } m \in \mathbb{N}.$$

Moreover, by the orthogonality relations with respect to the axis  $I$ , we get also

$$\partial_t^l K^p \partial_t^m (K_s^p + \Lambda^p K^p) = \partial_t^l \Lambda^p \partial_t^m (K_s^p + \Lambda^p K^p) = 0,$$

that is,

$$\sum_{i=1}^3 \partial_t^l k^{pi} \partial_t^m (k_s^{pi} + \lambda^{pi} k^{pi}) = \sum_{i=1}^3 \partial_t^l \lambda^{pi} \partial_t^m (k_s^{pi} + \lambda^{pi} k^{pi}) = 0 \text{ for every } l, m \in \mathbb{N}. \quad (3.5)$$

**Remark 3.2.** By the previous computations, for every solution in Definitions 2.10 or 2.11, at  $t > 0$  the curvature at the end-points and the sum of the three curvatures at every 3-point have to be zero and the same for the functions  $\lambda$ .

Then, a necessary condition for the maps  $\gamma^i$  to be  $C^2$  in space and  $C^1$  in time till the whole *parabolic boundary* (given by  $[0, 1] \times \{0\} \cup \{0, 1\} \times [0, T]$  in Definition 2.10 and  $[0, 1] \times \{0\} \cup \{0, 1\} \times [0, T]$  or  $[0, 1] \times \{0\} \cup \{0\} \times [0, T]$  in Definition 2.11), is that these conditions are satisfied also by the initial regular network  $\mathbb{S}_0$ , for some functions  $\lambda_0$  (see Remark 2.12) extending continuously the functions  $\lambda$  which are actually defined only for  $t > 0$ . That is, for the initial regular network  $\mathbb{S}_0$ , there must hold

$$(k\nu + \lambda_0\tau)(P^r) = 0, \text{ for every } r \in \{1, 2, \dots, l\}$$

and

$$(k^{pi}\nu^{pi} + \lambda_0^{pi}\tau^{pi})(O^p) = (k^{pj}\nu^{pj} + \lambda_0^{pj}\tau^{pj})(O^p), \text{ for every } i, j \in \{1, 2, 3\}.$$

In particular, for the initial network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i(I_i)$ , the curvature at the end-points and the sum of the three curvatures at every 3-point have to be zero.

These conditions on the curvatures of  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i(I_i)$  are clearly *geometric*, independent of the parametrizations of the curves  $\sigma^i$  but intrinsic to the set  $\mathbb{S}_0$  and they are not satisfied by a generic regular,  $C^2$  network

## 4 Short time existence I

In this section we start dealing with the problem of existence/uniqueness for short time of a solution of evolution Problem (2.3) for an initial regular network  $\mathbb{S}_0$ , with fixed end-points on the boundary of a smooth, convex, open set  $\Omega \subset \mathbb{R}^2$ .

We will establish a short time existence theorem of a curvature flow for a special class of  $C^{2+2\alpha}$  (second derivative in  $2\alpha$ -Hölder space, for some  $\alpha \in (0, 1/2)$ ) regular initial networks, satisfying some “compatibility conditions” at the end-points and at the 3-points. We analyze more general regular networks in Section 6.

Then we will discuss the problem of the uniqueness of the curvature flow of a network.

For sake of simplicity, we will deal in some detail with the case of the simplest possible network, a *triod* and we will explain how the same line works for a regular network with general structure.

**Definition 4.1.** A *triod*  $\mathbb{T} = \bigcup_{i=1}^3 \sigma^i([0, 1])$  is a network composed only of three  $C^1$  regular curves  $\sigma^i : [0, 1] \rightarrow \overline{\Omega}$  where  $\Omega$  is a smooth, convex, open subset of  $\mathbb{R}^2$ . These three curves intersect each other only at a single 3-point  $O$  and have the other three end-points coinciding with three distinct points  $P^i = \sigma^i(1) \in \overline{\Omega}$ .

An *open triod*  $\mathbb{T} = \bigcup_{i=1}^3 \sigma^i([0, 1))$  in  $\mathbb{R}^2$  is given by three  $C^1$  regular curves  $\sigma^i : [0, 1) \rightarrow \mathbb{R}^2$  which intersect each other only at a single 3-point  $O$  and each one of them is  $C^1$ -asymptotic to a halfline in  $\mathbb{R}^2$ , as  $x \rightarrow 1$ .

As before, the triod is regular if the exterior unit tangents of the three curves form angles of 120 degrees at the 3-point  $O$ .

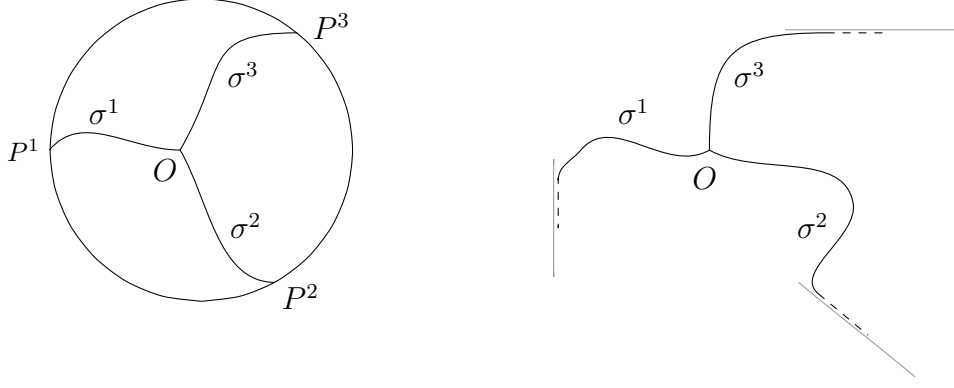


Figure 4: A regular triod on the left and an open regular triod on the right.

We restate Problem (2.3) for a triod.

The one parameter family of triods  $\mathbb{T}_t = \bigcup_{i=1}^3 \gamma^i([0, 1], t)$  is a flow by curvature in the time interval  $[0, T)$  of the initial, regular triod  $\mathbb{T}_0 = \bigcup_{i=1}^3 \sigma^i([0, 1])$  in a smooth convex, open set  $\Omega \subset \mathbb{R}^2$ , if the three maps  $\gamma^i : [0, 1] \times [0, T) \rightarrow \bar{\Omega}$  are continuous, there holds  $\gamma^i(x, 0) = \sigma^i(x)$  for every  $x \in [0, 1]$  and  $i \in \{1, 2, 3\}$  (initial data), they are at least  $C^2$  in space and  $C^1$  in time in  $[0, 1] \times (0, T)$  and satisfy the following system of conditions for every  $x \in [0, 1]$ ,  $t \in (0, T)$ ,  $i \in \{1, 2, 3\}$ ,

$$\left\{ \begin{array}{ll} \gamma_x^i(x, t) \neq 0 & \text{regularity} \\ \gamma^i(1, t) = P^i & \text{fixed end-points condition} \\ \sum_{i=1}^3 \frac{\gamma_x^i(0, t)}{|\gamma_x^i(0, t)|} = 0 & \text{angles of 120 degrees} \\ \gamma_t^i = k^i \nu^i + \lambda^i \tau^i & \text{for some continuous functions } \lambda^i \quad \text{motion by curvature} \end{array} \right. \quad (4.1)$$

In order to have a short time existence theorem, we make a special choice for the functions  $\lambda^i$ , by considering the system of quasilinear PDE's:

$$\left\{ \begin{array}{ll} \gamma_x^i(x, t) \neq 0 & \text{regularity} \\ \gamma^i(1, t) = P^i & \text{fixed end-points condition} \\ \gamma^i(x, 0) = \sigma^i(x) & \text{initial data} \\ \sum_{i=1}^3 \frac{\gamma_x^i(0, t)}{|\gamma_x^i(0, t)|} = 0 & \text{angles of 120 degrees} \\ \gamma_t^i(x, t) = \frac{\gamma_{xx}^i(x, t)}{|\gamma_x^i(x, t)|^2} & \text{motion by curvature} \end{array} \right. \quad (4.2)$$

where we substituted  $\gamma_t^i = k^i \nu^i + \lambda^i \tau^i$  with  $\gamma_t^i = \frac{\gamma_{xx}^i}{|\gamma_x^i|^2}$  for every  $x \in [0, 1]$ ,  $t \in [0, T)$  and  $i \in \{1, 2, 3\}$  (see Remark 2.8).

By means of a method of Bronsard and Reitich in [14], based on Solonnikov theory [78] (see also [52]), as the system (4.2) satisfies that the so-called *complementary conditions* (see [78, p. 11]), which are a sort of algebraic relations between the evolution equation and the boundary constraints at the 3-point and at the end-points of the triod (see [14, Section 3], for more detail), there exists a unique solution  $\gamma^i \in C^{2+2\alpha, 1+\alpha}([0, 1] \times [0, T))$  of system (4.2), for some maximal time  $T > 0$ , given any initial regular  $C^{2+2\alpha}$  triod  $\mathbb{T}_0 = \bigcup_{i=1}^3 \sigma^i([0, 1])$ , with  $\alpha \in (0, 1/2)$  provided it satisfies the so-called *compatibility conditions* of order 2.

**Definition 4.2.** We say that for system (4.2) the *compatibility conditions of order 2* are satisfied by the initial triod  $\mathbb{T}_0 = \bigcup_{i=1}^3 \sigma^i([0, 1])$  if at the end-points and at the 3-point, there hold all the relations on the space derivatives, up to second order, of the functions  $\sigma^i$  given by the boundary conditions and their time derivatives, assuming that the evolution equation holds also at such points.

Explicitly, the compatibility conditions of order 0 at the 3-point are

$$\sigma^i(0) = \sigma^j(0) \quad \text{for every } i, j \in \{1, 2, 3\}$$



and

$$\sigma^i(1) = P^i \quad \text{for every } i \in \{1, 2, 3\},$$

that is, simply the concurrency and fixed end-points conditions.

The compatibility condition of order 1 is given by

$$\sum_{i=1}^3 \frac{\sigma_x^i(0)}{|\sigma_x^i(0)|} = 0,$$

that is, the 120 degrees condition at the 3-point.

To get the second order conditions, one has to differentiate in time the first ones, getting

$$\frac{\sigma_{xx}^i(1)}{|\sigma_x^i(1)|^2} = 0 \quad \text{for every } i \in \{1, 2, 3\},$$

and

$$\frac{\sigma_{xx}^i(0)}{|\sigma_x^i(0)|^2} = \frac{\sigma_{xx}^j(0)}{|\sigma_x^j(0)|^2} \quad \text{for every } i, j \in \{1, 2, 3\}.$$

**Theorem 4.3** (Bronsard and Reitich [14]). *For any initial, regular  $C^{2+2\alpha}$  triod  $\mathbb{T}_0 = \bigcup_{i=1}^3 \sigma^i([0, 1])$ , with  $\alpha \in (0, 1/2)$ , satisfying the compatibility conditions of order 2, there exists a unique solution in  $C^{2+2\alpha, 1+\alpha}([0, 1] \times [0, T))$  of system (4.2). Moreover, every triod  $\mathbb{T}_t = \bigcup_{i=1}^3 \gamma^i([0, 1], t)$  satisfies the compatibility conditions of order 2.*

*Remark 4.4.* Actually, in [14] the authors do not consider exactly Problem 4.1, but the analogous “Neumann problem”. That is, they assign an angle conditions at the 3-point and require that the end-points of the three curves meet the boundary of  $\Omega$  with a prescribed angle (respectively, 120 and 90 degrees in the case explicitly proved in detail).

A solution of system (4.2) clearly provides a solution to Problem (4.1).

**Theorem 4.5.** *For any initial, regular  $C^{2+2\alpha}$  triod  $\mathbb{T}_0 = \bigcup_{i=1}^3 \sigma^i([0, 1])$ , with  $\alpha \in (0, 1/2)$ , in a smooth, convex, open set  $\Omega \subset \mathbb{R}^2$ , satisfying the compatibility conditions of order 2, there exists a  $C^{2+2\alpha, 1+\alpha}([0, 1] \times [0, T))$  curvature flow of  $\mathbb{T}_0$  in a maximal positive time interval  $[0, T)$ . Moreover, every triod  $\mathbb{T}_t = \bigcup_{i=1}^3 \gamma^i([0, 1], t)$  satisfies the compatibility conditions of order 2.*

*Proof.* If  $\gamma^i \in C^{2+2\alpha, 1+\alpha}([0, 1] \times [0, T))$  is a solution of system (4.2), then it solves Problem (4.1) with

$$\lambda^i(x, t) = \frac{\langle \gamma_{xx}^i(x, t) | \tau^i(x, t) \rangle}{|\gamma_x^i(x, t)|^2} = \frac{\langle \gamma_{xx}^i(x, t) | \gamma_x^i(x, t) \rangle}{|\gamma_x^i(x, t)|^3}.$$

Indeed, it follows immediately by the regularity properties of this flow that the relative functions  $\lambda^i$  belong to the parabolic Hölder space  $C^{2\alpha, \alpha}([0, 1] \times [0, T))$  (hence, in  $C^\alpha([0, 1] \times [0, T))$ ), thus continuous and all the triods  $\mathbb{T}_t$  are in  $C^{2+2\alpha}$ , satisfying the compatibility conditions of order 2.

The property that these evolving triods are regular follows by the standard fact that the maps  $\gamma_x^i$  are continuous, belonging to  $C^{1+2\alpha, 1/2+\alpha}([0, 1] \times [0, T))$  (see [51, Section 8.8]), hence, being  $\sigma^i$  regular curves,  $\gamma_x^i(x, t) \neq 0$  continues to hold for every  $x \in [0, 1]$  and for some positive interval of time.

The fact that a curve cannot self-intersect or two curve cannot intersect each other can be ruled out by noticing that such intersection cannot happen at the 3-point by geometric reasons, as the curvature is locally bounded and the curves are regular, then it is well known for the motion by curvature that strong maximum principle prevents such intersections for the flow of two embedded curves (or two distinct parts of the same curve). A similar argument and again the strong maximum principle also prevent that a curve “hits” the boundary of  $\Omega$  at a point different from a fixed end-point of the triod.  $\square$

The method of Bronsard and Reitich extends to the case of an initial regular network  $\mathbb{S}_0$  in a smooth, convex, open set  $\Omega \subset \mathbb{R}^2$ , with several 3-points and end-points. Indeed, as we said, such method relies on the uniform parabolicity of the system (which is the same) and on the fact that the complementary and compatibility conditions are satisfied.

**Definition 4.6.** We say that a regular  $C^2$  network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$  is *2-compatible* if the maps  $\sigma^i$  satisfy the *compatibility conditions of order 2* for system (4.3), that is,  $\sigma_{xx}^i = 0$  at every end-point and

$$\frac{\sigma_{xx}^{pi}(O^p)}{|\sigma_x^{pi}(O^p)|^2} = \frac{\sigma_{xx}^{pj}(O^p)}{|\sigma_x^{pj}(O^p)|^2}$$

for every pair of curves  $\sigma^{pi}$  and  $\sigma^{pj}$  concurring at any 3-point  $O^p$  (where we abused a little the notation like in Definition 2.10).

**Theorem 4.7.** For any initial, regular  $C^{2+2\alpha}$  network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$ , with  $\alpha \in (0, 1/2)$ , which is 2-compatible, there exists a unique solution in  $C^{2+2\alpha, 1+\alpha}([0, 1] \times [0, T])$  of the following quasilinear system of PDE's

$$\left\{ \begin{array}{ll} \gamma_x^i(x, t) \neq 0 & \text{regularity} \\ \gamma^r(1, t) = P^r & \text{with } 0 \leq r \leq l \quad \text{fixed end-points condition} \\ \gamma^i(x, 0) = \sigma^i(x) & \text{initial data} \\ \sum_{j=1}^3 \frac{\gamma_{xx}^{pj}(O^p, t)}{|\gamma_x^{pj}(O^p, t)|^2} = 0 & \text{at every 3-point } O^p \quad \text{angles of 120 degrees} \\ \gamma_t^i(x, t) = \frac{\gamma_{xx}^i(x, t)}{|\gamma_x^i(x, t)|^2} & \text{motion by curvature} \end{array} \right. \quad (4.3)$$

(where we used the notation of Definition 2.10) for every  $x \in [0, 1]$ ,  $t \in [0, T)$  and  $i \in \{1, 2, \dots, n\}$ , in a maximal positive time interval  $[0, T)$ .

Moreover, every network  $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i([0, 1], t)$  is 2-compatible.

As before, a solution of system (4.3) provides a solution to Problem (2.3), since the same geometric considerations in the proof of Theorem 4.5 hold also in this general case.

**Theorem 4.8.** For any initial, regular  $C^{2+2\alpha}$  network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$ , with  $\alpha \in (0, 1/2)$ , in a smooth, convex, open set  $\Omega \subset \mathbb{R}^2$ , which is 2-compatible, there exists a  $C^{2+2\alpha, 1+\alpha}([0, 1] \times [0, T])$  curvature flow of  $\mathbb{S}_0$  in a maximal positive time interval  $[0, T)$ .

Moreover, every network  $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i([0, 1], t)$  is 2-compatible.

We recall that a special curvature flow of  $\mathbb{S}_0$  is a curvature flow  $\gamma^i$  for the initial, regular  $C^2$  network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$  which satisfies  $\gamma_t^i = \frac{\gamma_{xx}^i}{|\gamma_x^i|^2}$  for every  $t > 0$  (see Definition 2.9).

By the very definition, every network of a special curvature flow is 2-compatible, for  $t > 0$ . Clearly, the solution given by Theorem 4.7 (which is the curvature flow mentioned in Theorem 4.8) is a special curvature flow.

**Remark 4.9.** Notice that if we have a  $C^{2,1}$  curvature flow  $\mathbb{S}_t$ , it is not necessarily 2-compatible for every time. It only have to satisfy  $k\nu + \lambda\tau = 0$  at every end-point and

$$(k^{pi}\nu^{pi} + \lambda^{pi}\tau^{pi})(O^p) = (k^{pj}\nu^{pj} + \lambda^{pj}\tau^{pj})(O^p), \text{ for every } i, j \in \{1, 2, 3\}$$

at every 3-point  $O^p$  (see Remark 3.2).

These relations implies anyway that for every evolving network  $\mathbb{S}_t$  the curvature is zero at every end-point and the sum of the three curvatures at every 3-point is zero. We see now that this implies that reparametrizing  $\mathbb{S}_t$  by a  $C^\infty$  map we obtain a 2-compatible network.

**Definition 4.10.** We say that a regular  $C^2$  network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$  is *geometrically 2-compatible* if it admits a regular reparametrization by a  $C^\infty$  map such that it becomes 2-compatible.

By this definition, it is trivial that the property to be geometrically 2-compatible is invariant by reparametrization of the curves of a network. Moreover, it is a geometric property of a network since it involves only the curvature, by the following lemma.

**Lemma 4.11.** If for a regular  $C^2$  network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$  the curvature is zero at every end-point and the sum of the three curvatures at every 3-point is zero, then  $\mathbb{S}_0$  is geometrically 2-compatible.

*Proof.* We look for some  $C^\infty$  maps  $\theta^i : [0, 1] \rightarrow [0, 1]$ , with  $\theta_x^i(x) \neq 0$  for every  $x \in [0, 1]$  and  $\theta^i(0) = 0$ ,  $\theta^i(1) = 1$  such that the reparametrized curves  $\tilde{\sigma}^i = \sigma^i \circ \theta^i$  satisfy

$$\frac{\tilde{\sigma}_{xx}^i}{|\tilde{\sigma}_x^i|^2} = \frac{\tilde{\sigma}_{xx}^j}{|\tilde{\sigma}_x^j|^2}$$

for every pair of concurring curves  $\tilde{\sigma}^i$  and  $\tilde{\sigma}^j$  at any 3-point and  $\tilde{\sigma}_{xx}^i = 0$  at every end-point of the network. Setting  $\tilde{\lambda}_0^i = \frac{\langle \tilde{\sigma}_{xx}^i | \tilde{\sigma}_x^i \rangle}{|\tilde{\sigma}_x^i|^3}$  this means

$$\tilde{k}^i \tilde{\nu}^i + \tilde{\lambda}_0^i \tilde{\tau}^i = \tilde{k}^j \tilde{\nu}^j + \tilde{\lambda}_0^j \tilde{\tau}^j$$

for every pair of concurring curves  $\tilde{\sigma}^i$  and  $\tilde{\sigma}^j$  at any 3-point and  $\tilde{k}^i \tilde{\nu}^i + \tilde{\lambda}_0^i \tilde{\tau}^i = 0$  at every end-point of the network. Since the curvature is invariant by reparametrization, by means of computations of Section 3 and the hypotheses on the curvature, these two conditions are satisfied if and only if  $\tilde{\lambda}_0^i = 0$  at every end-point of the network and

$$\tilde{\lambda}_0^i = \frac{k^{i-1} - k^{i+1}}{\sqrt{3}}$$

at every 3-point of the network, for  $i \in \{1, 2, 3\}$  (modulus 3).

Hence, we only need to find  $C^\infty$  reparametrizations  $\theta^i$  such that at the borders of  $[0, 1]$  the values of  $\tilde{\lambda}_0^i = \frac{\langle \tilde{\sigma}_{xx}^i | \tilde{\sigma}_x^i \rangle}{|\tilde{\sigma}_x^i|^3}$  are given by these relations. This can be easily done since at the borders of the interval  $[0, 1]$  we have  $\theta^i(0) = 0$  and  $\theta^i(1) = 1$ , hence

$$\tilde{\lambda}_0^i = \frac{\langle \tilde{\sigma}_{xx}^i | \tilde{\sigma}_x^i \rangle}{|\tilde{\sigma}_x^i|^3} = -\partial_x \frac{1}{|\tilde{\sigma}_x^i|} = -\partial_x \frac{1}{|\sigma_x^i \circ \theta^i| \theta_x^i} = \frac{\langle \sigma_{xx}^i | \sigma_x^i \rangle}{|\sigma_x^i|^3} + \frac{\theta_{xx}^i}{|\sigma_x^i| |\theta_x^i|^2} = \lambda_0^i + \frac{\theta_{xx}^i}{|\sigma_x^i| |\theta_x^i|^2}$$

where  $\lambda_0^i = \frac{\langle \sigma_{xx}^i | \sigma_x^i \rangle}{|\sigma_x^i|^3}$ , then we can simply choose any  $C^\infty$  functions  $\theta^i$  with  $\theta_x^i(0) = \theta_x^i(1) = 1$ ,  $\theta_{xx}^i = -\lambda_0^i |\sigma_x^i| |\theta_x^i|^2$  at every end-point and

$$\theta_{xx}^i = \left( \frac{k^{i-1} - k^{i+1}}{\sqrt{3}} - \lambda_0^i \right) |\sigma_x^i| |\theta_x^i|^2$$

at every 3-point of the network (for instance, one can use a polynomial function). It follows that the reparametrized network  $\tilde{\mathbb{S}}_0 = \bigcup_{i=1}^n (\sigma^i \circ \theta^i)([0, 1])$  is 2-compatible.  $\square$

By this lemma and Remark 4.9, we immediately have the following proposition.

**Proposition 4.12.** *Given a curvature flow  $\mathbb{S}_t$  of an initial regular  $C^2$  network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$ , all the networks  $\mathbb{S}_t$ , for  $t > 0$ , are geometrically 2-compatible.*

We now extend the short time existence result to regular,  $C^{2+2\alpha}$  initial networks which are geometrically 2-compatible.

**Theorem 4.13.** *For any initial regular  $C^{2+2\alpha}$  network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$  which is geometrically 2-compatible, with  $\alpha \in (0, 1/2)$ , in a smooth, convex, open set  $\Omega \subset \mathbb{R}^2$ , there exists a curvature flow which is in  $C^{2+2\alpha, 1+\alpha}([0, 1] \times [0, T))$  for a maximal positive time interval  $[0, T)$ .*

*Proof.* By the hypothesis, we can reparametrize the network  $\mathbb{S}_0$  with some  $C^\infty$  maps  $\theta^i$  in order it is 2-compatible. Clearly, if the network  $\mathbb{S}_0$  belongs to  $C^{2+2\alpha}$  the reparametrized one  $\tilde{\mathbb{S}}_0$  is still in  $C^{2+2\alpha}$ , hence, we can apply Theorem 4.8, to get the unique special curvature flow  $\tilde{\gamma}^i$  for  $\tilde{\mathbb{S}}_0 = \bigcup_{i=1}^n \tilde{\sigma}^i([0, 1]) = \bigcup_{i=1}^n (\sigma^i \circ \theta^i)([0, 1])$  which is in  $C^{2+2\alpha, 1+\alpha}([0, 1] \times [0, T))$  for a maximal positive time interval  $[0, T)$ . Moreover, every network  $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i([0, 1], t)$  is 2-compatible.

If now we consider the maps  $\gamma^i$  given by  $\gamma^i(x, t) = \tilde{\gamma}^i([\theta^i]^{-1}(x), t)$ , we have that they still belong to  $C^{2+2\alpha, 1+\alpha}([0, 1] \times [0, T))$  (as the maps  $[\theta^i]^{-1}$  are in  $C^\infty$ ),  $\gamma^i(\cdot, 0) = \sigma^i$  and

$$\begin{aligned} \gamma_t^i(x, t) &= \partial_t [\tilde{\gamma}^i([\theta^i]^{-1}(x), t)] \\ &= \tilde{\gamma}_t^i([\theta^i]^{-1}(x), t) \\ &= \tilde{k}^i([\theta^i]^{-1}(x), t) + \tilde{\lambda}^i([\theta^i]^{-1}(x), t) \tilde{\tau}^i([\theta^i]^{-1}(x), t) \\ &= \underline{k}^i(x, t) + \underline{\lambda}^i(x, t), \end{aligned}$$

with  $\underline{\lambda}^i(x, t) = \tilde{\lambda}^i([\theta^i]^{-1}(x), t) \tilde{\tau}^i([\theta^i]^{-1}(x), t)$ . Hence,  $\gamma^i$  is a flow by curvature of the network  $\mathbb{S}_0$  in  $C^{2+2\alpha, 1+\alpha}([0, 1] \times [0, T))$   $\square$

Clearly, it would be desirable to have an existence result for the flow on an initial  $C^{2+2\alpha}$  network which is not necessarily geometrically 2-compatible or simply a  $C^2$  network. We will try to address this problem in Section 6.

Anyway, suppose that the (only)  $C^2$  network  $\mathbb{S}_0$  is geometrically 2-compatible and we can find some (only)  $C^2$  regular reparametrization  $\varphi^i$  turning it in a  $C^{2+2\alpha}$  network, then, being this latter still geometrically 2-compatible we have a curvature flow by the previous theorem. Hence, composing this flow with the maps  $[\varphi^i]^{-1}$ , which are in  $C^2$ , we get a curvature flow for  $\mathbb{S}_0$  (this situation happens, for instance, considering a  $C^{2+2\alpha}$  network and reparametrizing it with maps which are  $C^2$  but not  $C^{2+2\alpha}$ , obtaining a  $C^2$ -only network  $\mathbb{S}_0$ ).

This fact is related to the geometric nature of this evolution problem, indeed, in general, given any curvature flow  $\gamma^i$  of an initial network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$ , setting  $\tilde{\sigma}^i = \sigma^i \circ \theta^i$  for some orientation preserving  $C^2$  functions  $\theta^i : [0, 1] \rightarrow [0, 1]$ , with  $\theta_x(x) \neq 0$  for every  $x \in [0, 1]$ ,  $\theta^i(0) = 0$  and  $\theta^i(1) = 1$ , we have, setting  $\tilde{\gamma}^i(x, t) = \gamma^i(\theta^i(x), t)$ ,

$$\tilde{\gamma}_t^i(x, t) = \partial_t[\gamma^i(\theta^i(x), t)] = \gamma_t^i(\theta^i(x), t) = \underline{k}^i(\theta^i(x), t) + \underline{\lambda}^i(\theta^i(x), t) = \tilde{k}^i(x, t) + \tilde{\lambda}^i(x, t),$$

with  $\tilde{\lambda}^i(x, t) = \underline{\lambda}^i(\theta^i(x), t)$ . Hence,  $\tilde{\gamma}^i = \gamma^i \circ \theta^i$  is a flow by curvature of the network  $\tilde{\mathbb{S}}_0 = \bigcup_{i=1}^n \tilde{\sigma}^i([0, 1])$  which is nothing more than a  $C^2$  reparametrization of the network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$ . It follows easily that if  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma_0^i([0, 1])$  and  $\tilde{\mathbb{S}}_0 = \bigcup_{i=1}^n \xi_0^i([0, 1])$  describe the same initial regular  $C^2$  network parametrized in two different ways, all the possible curvature flows of  $\tilde{\mathbb{S}}_0$  can be obtained by reparametrizations of the curvature flows of  $\mathbb{S}_0$  and viceversa.

Even more in general, considering the time-depending reparametrizations  $\tilde{\gamma}^i(x, t) = \gamma^i(\varphi^i(x, t), t)$  with some maps  $\varphi^i : [0, 1] \times [0, T) \rightarrow [0, 1]$  in  $C^0([0, 1] \times [0, T)) \cap C^2([0, 1] \times (0, T))$  such that  $\varphi^i(0, t) = 0$ ,  $\varphi^i(1, t) = 1$  and  $\varphi_x(x, t) \neq 0$  for every  $(x, t) \in [0, 1] \times [0, T)$ , we compute

$$\begin{aligned} \tilde{\gamma}_t^i(x, t) &= \partial_t[\gamma^i(\varphi^i(x, t), t)] \\ &= \gamma_x^i(\varphi^i(x, t), t) \varphi_t^i(x, t) + \gamma_t^i(\varphi^i(x, t), t) \\ &= \gamma_x^i(\varphi^i(x, t), t) \varphi_t^i(x, t) + \underline{k}^i(\varphi^i(x, t), t) + \underline{\lambda}^i(\varphi^i(x, t), t) \\ &= \tilde{k}^i(x, t) + \tilde{\lambda}^i(x, t), \end{aligned}$$

with  $\tilde{\lambda}^i(x, t) = \underline{\lambda}^i(\varphi^i(x, t), t) + \gamma_x^i(\varphi^i(x, t), t) \varphi_t^i(x, t)$ . Hence, the reparametrized evolving network composed by the curves  $\tilde{\gamma}^i$  is a curvature flow for the initial network  $\tilde{\mathbb{S}}_0 = \bigcup_{i=1}^n (\sigma^i \circ \varphi^i(\cdot, 0))([0, 1])$ . In particular, choosing special maps  $\varphi^i$  such that  $\varphi^i(x, 0) = x$  also holds, we have  $\tilde{\gamma}^i(x, 0) = \gamma^i(x, 0) = \sigma^i(x)$ , hence,  $\tilde{\gamma}^i$  is another curvature flow for the initial network  $\mathbb{S}_0 = \bigcup_{i=1}^n (\sigma^i)([0, 1])$ .

*Remark 4.14.* All this discussion suggests that the natural concept of uniqueness for the curvature flow of an initial network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$ , in our framework, is to consider uniqueness up to (dynamic)  $C^2$  reparametrization of the curves of the network, we will get back to this at the end of this section. Notice, moreover, that from what we saw above, we could also have considered networks simply as sets, forgetting their parametrization, and their curvature flows as flows of sets that could be parametrized in order to satisfy Definition 2.10.

We discuss now the higher regularity of the flow when the initial network is  $C^\infty$ .

**Definition 4.15.** We say that for system (4.3) the *compatibility conditions of every order* are satisfied by an initial regular  $C^\infty$  network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$  and we call such a network *smooth*, if at every end-points and every 3-point, there hold all the relations on the space derivatives of the functions  $\sigma^i$ , obtained repeatedly differentiating in time the boundary conditions and using the evolution equation  $\gamma_t^i(x, t) = \frac{\gamma_{xx}^i(x, t)}{|\gamma_x^i(x, t)|^2}$  to substitute time derivatives with space derivatives.

We say that a  $C^\infty$  flow by curvature  $\mathbb{S}_t$  is *smooth* if all the networks  $\mathbb{S}_t$  are *smooth*.

It is immediate by this definition that every network  $\mathbb{S}_t$  of a  $C^\infty$  special curvature flow of an initial regular network  $\mathbb{S}_0$ , is smooth for every  $t > 0$ .

*Remark 4.16.* Notice that (compare with Remark 4.9) for the curvature flow of a network being smooth is quite more than being simply  $C^\infty$  up to the parabolic boundary. Anyway, analogously as before (Proposition 4.12), every network of a  $C^\infty$  curvature flow can be reparametrized to be smooth.

If we assume that the initial regular network is smooth, we have the following higher regularity result.

**Theorem 4.17.** *For any initial smooth, regular network  $\mathbb{S}_0$  in a smooth, convex, open set  $\Omega \subset \mathbb{R}^2$  there exists a unique  $C^\infty$  solution of system (4.3) in a maximal time interval  $[0, T)$ .*

*Proof.* Since the initial network  $\mathbb{S}_0$  satisfies the compatibility condition at every order, the method of Bronsard and Reitich actually provides, for every  $n \in \mathbb{N}$ , a unique solution in  $C^{2n+2\alpha, n+\alpha}([0, 1] \times [0, T_n])$  of system (4.3) satisfying the compatibility conditions of order  $0, 1, \dots, 2n$  at every time.

So, if we have a solution  $\gamma^i \in C^{2n+2\alpha, n+\alpha}([0, 1] \times [0, T_n])$  for  $n \geq 1$ , then the functions  $\gamma_x^i$  belong to  $C^{2n-1+2\alpha, n-1/2+\alpha}([0, 1] \times [0, T_n])$  (see [51, Section 8.8]). Considering the parabolic system satisfied by  $v^i(x, t) = \gamma_t^i(x, t)$  (see [63, p. 250]), by Solonnikov results in [78],  $v^i = \gamma_t^i$  belongs to  $C^{2n+2\alpha, n+\alpha}([0, 1] \times [0, T_n])$  and since  $\gamma_{xx}^i = \gamma_t^i |\gamma_x^i|^2$  with  $|\gamma_x^i|^2 \in C^{2n-1+2\alpha, n-1/2+\alpha}([0, 1] \times [0, T_n])$ , we get also  $\gamma_{xx}^i \in C^{2n-1+2\alpha, n-1/2+\alpha}([0, 1] \times [0, T_n])$ .

Following [57], we can then conclude that  $\gamma^i \in C^{2n+1+2\alpha, n+1/2+\alpha}([0, 1] \times [0, T_n])$ .

Iterating this argument, we see that  $\gamma^i \in C^\infty([0, 1] \times [0, T_n])$ , moreover, since for every  $n \in \mathbb{N}$  the solution obtained via the method of Bronsard and Reitich is unique, it must coincide with  $\gamma^i$  and we can choose all the  $T_n$  to be the same positive value  $T$ .

It follows that the solution is in  $C^\infty$  till the parabolic boundary, hence, all the compatibility conditions are satisfied at every time  $t \in [0, T)$ .  $\square$

As a consequence, we have the following theorem.

**Theorem 4.18.** *For any initial smooth, regular network  $\mathbb{S}_0$  in a smooth, convex, open set  $\Omega \subset \mathbb{R}^2$  there exists a smooth special curvature flow of  $\mathbb{S}_0$  in a maximal positive time interval  $[0, T)$ .*

Also for  $C^\infty$  networks one can introduce the concept of *geometrically smoothness*. A network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$  is geometrically smooth if it can be reparametrized to be smooth.

*Remark 4.19.* By arguments similar to the ones of Lemma 4.11, it can be shown (by means of the computations of the next section) that, like for geometrical 2-compatibility, this property depends only on (some relations on) the curvature and its derivatives at the end-points and at the 3-points of a  $C^\infty$  network (see [63] for more details), that is, geometrical smoothness is again a geometric property (obviously invariant by  $C^\infty$  reparametrizations, by the definition).

Moreover, analogously as before (see Proposition 4.12), every  $C^\infty$  curvature flow of an initial regular network  $\mathbb{S}_0$  is actually composed of *geometrically smooth* networks  $\mathbb{S}_t$ , for every  $t > 0$ .

The following analogous short time existence theorem holds, essentially with the same proof of Theorem 4.13.

**Theorem 4.20.** *For any initial geometrically smooth, regular network  $\mathbb{S}_0$  in a smooth, convex, open set  $\Omega \subset \mathbb{R}^2$  there exists a  $C^\infty$  curvature flow of  $\mathbb{S}_0$  in a maximal positive time interval  $[0, T)$ .*

We want to discuss now the concept of uniqueness of the flow. Even if the solution of system (4.3) is unique we have seen that there are anyway several solutions of Problem (2.3) for the same initial data, by dynamically reparametrizing it with maps  $\varphi^i : [0, 1] \times [0, T) \rightarrow [0, 1]$  in  $C^0([0, 1] \times [0, T)) \cap C^2([0, 1] \times (0, T))$  such that  $\varphi^i(0, t) = 0$ ,  $\varphi^i(1, t) = 1$ ,  $\varphi^i(x, 0) = x$  and  $\varphi_x(x, t) \neq 0$  for every  $(x, t) \in [0, 1] \times [0, T)$ . Hence, we consider uniqueness of the flow of an initial network, up to these reparametrizations.

**Definition 4.21.** We say that the curvature flow of an initial  $C^2$  network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$  is *geometrically unique* (in some regularity class), if all the curvature flows satisfying Definition 2.10 can be obtained each other by means of time-depending reparametrizations.

To be precise, this means that if  $\mathbb{S}_t$  and  $\tilde{\mathbb{S}}_t$  are two curvature flows of  $\mathbb{S}_0$ , described by some maps  $\gamma^i$  and  $\tilde{\gamma}^i$ , there exists a family of maps  $\varphi^i : [0, 1] \times [0, T) \rightarrow [0, 1]$  in  $C^0([0, 1] \times [0, T)) \cap C^2([0, 1] \times (0, T))$  such that  $\varphi^i(0, t) = 0$ ,  $\varphi^i(1, t) = 1$ ,  $\varphi^i(x, 0) = x$ ,  $\varphi_x^i(x, t) \neq 0$  and  $\tilde{\gamma}^i(x, t) = \gamma^i(\varphi^i(x, t), t)$  for every  $(x, t) \in [0, 1] \times [0, T)$ .

It is obvious that if there is geometric uniqueness, any curvature flow gives a unique evolved network as a subset of  $\mathbb{R}^2$ , for every time  $t \in [0, T)$ , which is still the same set also if we change the parametrization of the initial network by the previous discussion.

Unfortunately, at the moment, the problem of geometric uniqueness of the curvature flow of a regular network in the class  $C^{2,1}$  is open (even if the initial network is smooth). It is quite natural to conjecture that it holds, but the only available partial result, up to our knowledge, is given by the following proposition, consequence of the uniqueness part of Theorem 4.7.

**Proposition 4.22.** *For any initial, regular  $C^{2+2\alpha}$ , with  $\alpha \in (0, 1/2)$ , network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$  which is geometrically 2-compatible, in a smooth, convex, open set  $\Omega \subset \mathbb{R}^2$ , there exists a geometrically unique solution  $\gamma^i$  of Problem (2.3) in the class of maps  $C^{2+2\alpha, 1+\alpha}([0, 1] \times [0, T))$  in a maximal positive time interval  $[0, T)$ .*

*Proof.* We showed the existence of a special solution  $\gamma^i$  in Theorem 4.13.

We first show that if  $\mathbb{S}_0$  satisfies the compatibility conditions of order 2 then the solution given by Theorem 4.7 is geometrically unique among the curvature flows in the class  $C^{2+2\alpha, 1+\alpha}([0, 1] \times [0, T))$ .

Suppose that  $\tilde{\gamma}^i : [0, 1] \times [0, T') \rightarrow \bar{\Omega}$  is another maximal solution in  $C^{2+2\alpha, 1+\alpha}([0, 1] \times [0, T'))$  satisfying  $\tilde{\gamma}_t^i = \tilde{k}^i \tilde{\nu}^i + \tilde{\lambda}^i \tilde{\tau}^i$  for some functions  $\tilde{\lambda}^i$  in  $C^{2\alpha}([0, 1] \times [0, T'))$ , we want to see that it coincides to  $\gamma^i$  up to a reparametrization of the curves  $\tilde{\gamma}^i(\cdot, t)$  for every  $t \in [0, \min\{T, T'\})$ . If we consider functions  $\varphi^i : [0, 1] \times [0, \min\{T, T'\}) \rightarrow [0, 1]$  belonging to  $C^{2+2\alpha, 1+\alpha}([0, 1] \times [0, \min\{T, T'\}))$  and the reparametrizations  $\bar{\gamma}^i(x, t) = \tilde{\gamma}^i(\varphi^i(x, t), t)$ , we have that  $\bar{\gamma}^i \in C^{2+2\alpha, 1+\alpha}([0, 1] \times [0, \min\{T, T'\}))$  and

$$\begin{aligned} \bar{\gamma}_t^i(x, t) &= \partial_t [\tilde{\gamma}^i(\varphi^i(x, t), t)] \\ &= \tilde{\gamma}_x^i(\varphi^i(x, t), t) \varphi_t^i(x, t) + \tilde{\gamma}_t^i(\varphi^i(x, t), t) \\ &= \tilde{\gamma}_x^i(\varphi^i(x, t), t) \varphi_t^i(x, t) + \tilde{k}^i(\varphi^i(x, t), t) + \tilde{\lambda}^i(\varphi^i(x, t), t) \\ &= \tilde{\gamma}_x^i(\varphi^i(x, t), t) \varphi_t^i(x, t) + \frac{\langle \tilde{\gamma}_{xx}^i(\varphi^i(x, t), t) \mid \tilde{\nu}^i(\varphi^i(x, t), t) \rangle}{|\tilde{\gamma}_x^i(\varphi^i(x, t), t)|^2} \tilde{\nu}^i(\varphi^i(x, t), t) \\ &\quad + \tilde{\lambda}^i(\varphi^i(x, t), t) \frac{\tilde{\gamma}_x^i(\varphi^i(x, t), t)}{|\tilde{\gamma}_x^i(\varphi^i(x, t), t)|}. \end{aligned}$$

We choose now maps  $\varphi^i \in C^{2+2\alpha, 1+\alpha}([0, 1] \times [0, T''))$  which are solutions for some positive interval of time  $[0, T'')$  of the following quasilinear PDE's

$$\varphi_t^i(x, t) = \frac{\langle \tilde{\gamma}_{xx}^i(\varphi^i(x, t), t) \mid \tilde{\gamma}_x^i(\varphi^i(x, t), t) \rangle}{|\tilde{\gamma}_x^i(\varphi^i(x, t), t)|^4} - \frac{\tilde{\lambda}^i(\varphi^i(x, t), t)}{|\tilde{\gamma}_x^i(\varphi^i(x, t), t)|} + \frac{\varphi_{xx}^i(x, t)}{|\tilde{\gamma}_x^i(\varphi^i(x, t), t)|^2 |\varphi_x^i(x, t)|^2}, \quad (4.4)$$

with  $\varphi^i(0, t) = 0$ ,  $\varphi^i(1, t) = 1$  and  $\varphi^i(x, 0) = x$  (hence,  $\bar{\gamma}^i(x, 0) = \gamma^i(x, 0) = \sigma^i(x)$ ). The existence of such solutions follows by standard theory of quasilinear parabolic equations (see [52, 56]), noticing that the initial data  $\varphi^i(x, 0) = x$  satisfies the compatibility conditions of order 2 for equation (4.4). Moreover, it is not difficult, by pushing a little the analysis, to show that  $\varphi_x(x, t) \neq 0$  and that  $T''$  can be taken equal to  $T'$ .

Then, it follows

$$\begin{aligned} \bar{\gamma}_t^i(x, t) &= \frac{\langle \tilde{\gamma}_{xx}^i(\varphi^i(x, t), t) \mid \tilde{\gamma}_x^i(\varphi^i(x, t), t) \rangle}{|\tilde{\gamma}_x^i(\varphi^i(x, t), t)|^4} \tilde{\gamma}_x^i(\varphi^i(x, t), t) + \frac{\varphi_{xx}^i(x, t) \tilde{\gamma}_x^i(\varphi^i(x, t), t)}{|\tilde{\gamma}_x^i(\varphi^i(x, t), t)|^2 |\varphi_x^i(x, t)|^2} \\ &\quad + \frac{\langle \tilde{\gamma}_{xx}^i(\varphi^i(x, t), t) \mid \tilde{\nu}^i(\varphi^i(x, t), t) \rangle}{|\tilde{\gamma}_x^i(\varphi^i(x, t), t)|^2} \tilde{\nu}^i(\varphi^i(x, t), t) \\ &= \frac{\langle \tilde{\gamma}_{xx}^i(\varphi^i(x, t), t) \mid \tilde{\tau}^i(\varphi^i(x, t), t) \rangle}{|\tilde{\gamma}_x^i(\varphi^i(x, t), t)|^2} \tilde{\tau}^i(\varphi^i(x, t), t) + \frac{\varphi_{xx}^i(x, t) \tilde{\gamma}_x^i(\varphi^i(x, t), t)}{|\tilde{\gamma}_x^i(\varphi^i(x, t), t)|^2 |\varphi_x^i(x, t)|^2} \\ &\quad + \frac{\langle \tilde{\gamma}_{xx}^i(\varphi^i(x, t), t) \mid \tilde{\nu}^i(\varphi^i(x, t), t) \rangle}{|\tilde{\gamma}_x^i(\varphi^i(x, t), t)|^2} \tilde{\nu}^i(\varphi^i(x, t), t) \\ &= \frac{\tilde{\gamma}_{xx}^i(\varphi^i(x, t), t)}{|\tilde{\gamma}_x^i(\varphi^i(x, t), t)|^2} + \frac{\varphi_{xx}^i(x, t) \tilde{\gamma}_x^i(\varphi^i(x, t), t)}{|\tilde{\gamma}_x^i(\varphi^i(x, t), t)|^2 |\varphi_x^i(x, t)|^2} \\ &= \frac{\bar{\gamma}_{xx}^i(x, t)}{|\bar{\gamma}_x^i(x, t)|^2}. \end{aligned}$$



We can then conclude that, by the uniqueness part of Theorem 4.7,  $\bar{\gamma}^i = \gamma^i$  for every  $i \in \{1, 2, \dots, n\}$ , hence  $\gamma^i(x, t) = \tilde{\gamma}^i(\varphi^i(x, t), t)$  in the time interval  $[0, \min\{T, T'\})$ . Since this “reparametrization relation” between any two maximal solutions of Problem (2.3) is symmetric (by means of the maps  $[\varphi^i]^{-1}$ ), it follows that  $T' = T$  and we are done.

Assume now that the network  $\mathbb{S}_0$  is only geometrically 2-compatible, then the proof of Theorem 4.13 gives a special solution  $\gamma^i$  given by  $\gamma^i(x, t) = \tilde{\gamma}^i([\theta^i]^{-1}(x), t)$  where  $\theta^i$  are smooth maps and  $\tilde{\gamma}^i$  is a special solution as above for the 2-compatible network  $\tilde{\mathbb{S}}_0 = \bigcup_{i=1}^n \tilde{\sigma}^i([0, 1]) = \bigcup_{i=1}^n (\sigma^i \circ \theta^i)([0, 1])$  which is in  $C^{2+2\alpha, 1+\alpha}([0, 1] \times [0, T])$  for a maximal positive time interval  $[0, T)$ .

Suppose that  $\bar{\gamma}^i : [0, 1] \times [0, T') \rightarrow \bar{\Omega}$  is another maximal flow for  $\mathbb{S}_0$  in  $C^{2+2\alpha, 1+\alpha}([0, 1] \times [0, T'))$  satisfying  $\bar{\gamma}_t^i = \bar{k}^i \bar{\nu}^i + \bar{\lambda}^i \bar{\tau}^i$  for some functions  $\bar{\lambda}^i$  in  $C^{2\alpha}([0, 1] \times [0, T'))$ . If we consider the maps  $\tilde{\bar{\gamma}}^i(x, t) = \bar{\gamma}^i(\theta^i(x), t)$  these gives a  $C^{2+2\alpha, 1+\alpha}([0, 1] \times [0, T'))$  curvature flow of the initial network  $\tilde{\mathbb{S}}_0$  which satisfies the compatibility conditions of order 2, hence, by the above argument,  $T' = T$  and the maps  $\tilde{\bar{\gamma}}^i$  and  $\tilde{\gamma}^i$  only differ by reparametrization by some maps  $\varphi^i \in C^{2+2\alpha, 1+\alpha}([0, 1] \times [0, T))$  with  $\varphi^i(x, 0) = x$ , that is

$$\tilde{\bar{\gamma}}^i(x, t) = \tilde{\gamma}^i(\varphi^i(x, t), t).$$

It follows that

$$\bar{\gamma}^i(x, t) = \tilde{\bar{\gamma}}^i([\theta^i]^{-1}(x), t) = \tilde{\gamma}^i(\varphi^i([\theta^i]^{-1}(x), t), t) = \gamma^i(\theta^i(\varphi^i([\theta^i]^{-1}(x), t)), t)$$

which shows that the two flows  $\bar{\gamma}^i$  and  $\gamma^i$  of the initial network  $\mathbb{S}_0$  coincide up to the time-dependent reparametrizations  $(x, t) \mapsto (\theta^i(\varphi^i([\theta^i]^{-1}(x), t)), t)$ .  $\square$

An immediate consequence is the following.

**Corollary 4.23.** *For any initial, regular geometrically smooth network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$  in a smooth, convex, open set  $\Omega \subset \mathbb{R}^2$ , there exists a geometrically unique solution of Problem (2.3) in the class of maps  $C^{2+2\alpha, 1+\alpha}([0, 1] \times [0, T))$  in a maximal positive time interval  $[0, T)$ . Moreover, such solution is  $C^\infty$ .*

*Remark 4.24.* Notice that it follows that any curvature flow as in the hypotheses of the above theorem and corollary is a reparametrization (of class  $C^{2+2\alpha, 1+\alpha}$  in the first case and  $C^\infty$  in the latter) of the special curvature flow given by Theorem 4.7 (which is  $C^\infty$  under the hypotheses of the corollary, by Theorem 4.17).

We conclude this section stating the following natural open problem, that we already mentioned, related to geometric uniqueness of the flow.

**Open Problem 4.25.** Show that for any initial, regular  $C^{2+2\alpha}$  network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$ , with  $\alpha \in (0, 1/2)$ , which is 2-compatible, the solution given by Theorem 4.8 is the *geometrically unique* curvature flow of  $\mathbb{S}_0$ . That is, the maps  $\gamma^i$  give the geometrically unique solution of Problem (2.3) in the class of continuous maps  $\gamma^i : [0, 1] \times [0, T) \rightarrow \mathbb{R}^2$  which are of class at least  $C^2$  in space and  $C^1$  in time in  $[0, 1] \times (0, T)$ .

The difficulty in getting such a uniqueness result is connected to the lack of some sort of (possibly geometric) maximum principle for this evolution problem.

## 5 Integral estimates

In this section we work out some integral estimates for a special flow by curvature of a smooth regular network. These estimates were previously proved for the case of the special curvature flow of a regular smooth triod with fixed end-points, in [63]. We now extend them to the case of a regular smooth network with “controlled” behavior of its end-points. An outline for such estimates with controlled behavior of the endpoints, for a general curvature flow appeared in [47, Section 7]. We advise the reader that when the computations are exactly the same we will refer directly to [63, Section 3], where it is possible to find every detail.

In all this section we will assume that the special flow by curvature is given by a  $C^\infty$  solution  $\gamma^i$  of system (4.3), that is, there holds

$$\gamma_t^i(x, t) = \frac{\gamma_{xx}^i(x, t)}{|\gamma_x^i(x, t)|^2},$$

(see Remark 2.8 and Definition 2.9 for the case of an initial  $C^2$  network). The estimates which only involve geometric quantities and do not involve the tangential speeds  $\lambda_i$  hold also for any smooth flow (the ones where we do not use the special form of the functions  $\lambda^i$  given by this equation). In order to use these estimates for a general smooth flow, because of geometric uniqueness (see Corollary 4.23 and Remark 4.24), one must reparametrize such a flow, preserving the boundary conditions (5.1) below, so it becomes special, then carry back the geometric (invariant by reparametrization) estimates to the original flow. Alternatively, one can also directly prove these estimates without reparametrizing first to a special flow, see [47, Section 7].

We will see that such special flow of a regular smooth network with “controlled” end-points exists smooth as long as the curvature stays bounded and none of the lengths of the curves goes to zero (Theorem 5.13).

We suppose that the special solution maps  $\gamma^i$  above exist and are of class  $C^\infty$  in the time interval  $[0, T)$  and that they describe the flow of a regular  $C^\infty$  network  $\mathbb{S}_t$  in  $\Omega$ , composed by  $n$  curves  $\gamma^i(\cdot, t) : [0, 1] \rightarrow \overline{\Omega}$  with  $m$  3-points  $O^1, O^2, \dots, O^m$  and  $l$  end-points  $P^1, P^2, \dots, P^l$ . We will assume that either such end-points are fixed or that there exist uniform (in time) constants  $C_j$ , for every  $j \in \mathbb{N}$ , such that

$$|\partial_s^j k(P^r, t)| + |\partial_s^j \lambda(P^r, t)| \leq C_j, \quad (5.1)$$

for every  $t \in [0, T)$  and  $r \in 1, 2, \dots, l$ .

The very first computation we are going to show is the evolution in time of the total length of a network under the curvature flow.

**Proposition 5.1.** *The time derivative of the measure  $ds$  on any curve  $\gamma^i$  of the network is given by the measure  $(\lambda_s^i - (k^i)^2) ds$ . As a consequence, we have*

$$\frac{dL^i(t)}{dt} = \lambda^i(1, t) - \lambda^i(0, t) - \int_{\gamma^i(\cdot, t)} (k^i)^2 ds$$

and

$$\frac{dL(t)}{dt} = \sum_{r=1}^l \lambda(P^r, t) - \int_{\mathbb{S}_t} k^2 ds,$$

where, with a little abuse of notation,  $\lambda(P^r, t)$  is the tangential velocity at the end-point  $P^r$  of the curve of the network getting at such point, for any  $r \in \{1, 2, \dots, l\}$ .

In particular, if the end-points  $P^r$  of the network are fixed during the evolution, we have

$$\frac{dL(t)}{dt} = - \int_{\mathbb{S}_t} k^2 ds, \quad (5.2)$$

thus, in such case, the total length  $L(t)$  is decreasing in time and uniformly bounded above by the length of the initial network  $\mathbb{S}_0$ .

*Proof.* The formula for the time derivative of the measure  $ds$  follows easily by the commutation formula (3.1). Then,

$$\frac{dL^i(t)}{dt} = \frac{d}{dt} \int_{\gamma^i(\cdot, t)} 1 ds = \int_{\gamma^i(\cdot, t)} (\lambda_s^i - (k^i)^2) ds = \lambda^i(1, t) - \lambda^i(0, t) - \int_{\gamma^i(\cdot, t)} (k^i)^2 ds.$$

Adding these relations for all the curves, the contributions of  $\lambda^{p_i}$  at every 3-point  $O^p$  vanish, by relation (3.3), and the formula of the statement follows. If the end-points are fixed all the terms  $\lambda(P^r, t)$  are zero and the last formula follows.  $\square$

The following notation will be very useful for the next computations in this section.

**Definition 5.2.** We will denote with  $\mathfrak{p}_\sigma(\partial_s^j \lambda, \partial_s^h k)$  a polynomial with constant coefficients in  $\lambda, \dots, \partial_s^j \lambda$  and  $k, \dots, \partial_s^h k$  such that every monomial it contains is of the form

$$C \prod_{l=0}^j (\partial_s^l \lambda)^{\alpha_l} \cdot \prod_{l=0}^h (\partial_s^l k)^{\beta_l} \quad \text{with} \quad \sum_{l=0}^j (l+1)\alpha_l + \sum_{l=0}^h (l+1)\beta_l = \sigma,$$

we will call  $\sigma$  the *geometric order* of  $\mathfrak{p}_\sigma$ .

Moreover, if one of the two arguments of  $\mathfrak{p}_\sigma$  does not appear, it means that the polynomial does not contain it, for instance,  $\mathfrak{p}_\sigma(\partial_s^h k)$  does not contain neither  $\lambda$  nor its derivatives.

We will denote with  $\mathfrak{q}_\sigma(\partial_t^j \lambda, \partial_s^h k)$  a polynomial as before in  $\lambda, \dots, \partial_t^j \lambda$  and  $k, \dots, \partial_s^h k$  such that all its monomials are of the form

$$C \prod_{l=0}^j (\partial_t^l \lambda)^{\alpha_l} \cdot \prod_{l=0}^h (\partial_s^l k)^{\beta_l} \quad \text{with} \quad \sum_{l=0}^j (2l+1)\alpha_l + \sum_{l=0}^h (l+1)\beta_l = \sigma.$$

Finally, when we will write  $\mathfrak{p}_\sigma(|\partial_s^j \lambda|, |\partial_s^h k|)$  (or  $\mathfrak{q}_\sigma(|\partial_t^j \lambda|, |\partial_s^h k|)$ ) we will mean a finite sum of terms like

$$C \prod_{l=0}^j |\partial_s^l \lambda|^{\alpha_l} \cdot \prod_{l=0}^h |\partial_s^l k|^{\beta_l} \quad \text{with} \quad \sum_{l=0}^j (l+1)\alpha_l + \sum_{l=0}^h (l+1)\beta_l = \sigma,$$

where  $C$  is a positive constant and the exponents  $\alpha_l, \beta_l$  are non negative *real* values (analogously for  $\mathfrak{q}_\sigma$ ).

Clearly we have  $\mathfrak{p}_\sigma(\partial_s^j \lambda, \partial_s^h k) \leq \mathfrak{p}_\sigma(|\partial_s^j \lambda|, |\partial_s^h k|)$ .

By means of the commutation rule (3.1), the relations in the next lemma are easily proved by induction (Lemmas 3.7 and 3.8 in [63]), starting from the relations in Section 3.

**Lemma 5.3.** *The following formulas hold for every curve of the evolving network  $\mathbb{S}_t$ :*

$$\begin{aligned} \partial_t \partial_s^j k &= \partial_s^{j+2} k + \lambda \partial_s^{j+1} k + \mathfrak{p}_{j+3}(\partial_s^j k) && \text{for every } j \in \mathbb{N}, \\ \partial_s^j k &= \partial_t^{j/2} k + \mathfrak{q}_{j+1}(\partial_t^{j/2-1} \lambda, \partial_s^{j-1} k) && \text{if } j \geq 2 \text{ is even,} \\ \partial_s^j k &= \partial_t^{(j-1)/2} k_s + \mathfrak{q}_{j+1}(\partial_t^{(j-3)/2} \lambda, \partial_s^{j-1} k) && \text{if } j \geq 1 \text{ is odd,} \\ \partial_t \partial_s^j \lambda &= \partial_s^{j+2} \lambda - \lambda \partial_s^{j+1} \lambda - 2k \partial_s^{j+1} k + \mathfrak{p}_{j+3}(\partial_s^j \lambda, \partial_s^j k) && \text{for every } j \in \mathbb{N}, \\ \partial_s^j \lambda &= \partial_t^{j/2} \lambda + \mathfrak{p}_{j+1}(\partial_s^{j-1} \lambda, \partial_s^{j-1} k) && \text{if } j \geq 2 \text{ is even,} \\ \partial_s^j \lambda &= \partial_t^{(j-1)/2} \lambda_s + \mathfrak{p}_{j+1}(\partial_s^{j-1} \lambda, \partial_s^{j-1} k) && \text{if } j \geq 1 \text{ is odd.} \end{aligned}$$

*Remark 5.4.* Notice that, by relations (3.4) at any 3-point  $O^p$  of the network there holds  $\partial_t^j \lambda^{pi} = (\mathbb{S} \partial_t^j \mathbb{K})^{pi}$ , that is, the time derivatives of  $\lambda^{pi}$  are expressible as time derivatives of the functions  $k^{pi}$ . Then, by using repeatedly such relation and the first formula of Lemma 5.3, we can express these latter as space derivatives of  $k^{pi}$ . Hence, we will have the relation

$$\sum_{i=1}^3 \mathfrak{q}_\sigma(\partial_t^j \lambda^{pi}, \partial_s^h k^{pi}) \Big|_{\text{at the 3-point } O^p} = \mathfrak{p}_\sigma(\partial_s^{\max\{2j,h\}} \mathbb{K}^p) \Big|_{\text{at the 3-point } O^p}$$

with the meaning that this last polynomial contains also product of derivatives of different  $k^{pi}$ 's, because of the action of the linear operator  $\mathbb{S}$ .

We will often make use of this identity in the computations in the sequel in the following form,

$$\sum_{i=1}^3 \mathfrak{q}_\sigma(\partial_t^j \lambda^{pi}, \partial_s^h k^{pi}) \Big|_{\text{at the 3-point } O^p} \leq \|\mathfrak{p}_\sigma(|\partial_s^{\max\{2j,h\}} k|)\|_{L^\infty}.$$

*Remark 5.5.* We state the following *calculus rules* which will be used extensively in the sequel,

$$\begin{aligned} \mathfrak{p}_\alpha(\partial_s^j \lambda, \partial_s^h k) \cdot \mathfrak{p}_\beta(\partial_s^l \lambda, \partial_s^m k) &= \mathfrak{p}_{\alpha+\beta}(\partial_s^{\max\{j,l\}} \lambda, \partial_s^{\max\{h,m\}} k), \\ \mathfrak{q}_\alpha(\partial_t^j \lambda, \partial_s^h k) \cdot \mathfrak{q}_\beta(\partial_t^l \lambda, \partial_s^m k) &= \mathfrak{q}_{\alpha+\beta}(\partial_t^{\max\{j,l\}} \lambda, \partial_s^{\max\{h,m\}} k). \end{aligned}$$

We already saw that the time derivatives of  $k$  and  $\lambda$  can be expressed in terms of space derivatives of  $k$  at any 3-point, the same holds for the space derivatives of  $\lambda$ , arguing by induction using the last two formulas in Lemma 5.3. Hence, it follows that

$$\begin{aligned} \partial_s^l \mathfrak{p}_\alpha(\partial_s^j \lambda, \partial_s^h k) &= \mathfrak{p}_{\alpha+l}(\partial_s^{j+l} \lambda, \partial_s^{h+l} k), & \partial_t^l \mathfrak{p}_\alpha(\partial_s^j \lambda, \partial_s^h k) &= \mathfrak{p}_{\alpha+2l}(\partial_s^{j+2l} \lambda, \partial_s^{h+2l} k) \\ \partial_t^l \mathfrak{q}_\alpha(\partial_t^j \lambda, \partial_s^h k) &= \mathfrak{q}_{\alpha+2l}(\partial_t^{j+2l} \lambda, \partial_s^{h+2l} k), & \mathfrak{q}_\alpha(\partial_t^j \lambda, \partial_s^h k) &= \mathfrak{p}_\alpha(\partial_s^{2j} \lambda, \partial_s^{\max\{h, 2j-1\}} k). \end{aligned}$$

We are now ready to compute, for  $j \in \mathbb{N}$ ,

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{S}_t} |\partial_s^j k|^2 ds &= 2 \int_{\mathbb{S}_t} \partial_s^j k \partial_t \partial_s^j k ds + \int_{\mathbb{S}_t} |\partial_s^j k|^2 (\lambda_s - k^2) ds \\
&= 2 \int_{\mathbb{S}_t} \partial_s^j k \partial_s^{j+2} k + \lambda \partial_s^{j+1} k \partial_s^j k + \mathfrak{p}_{j+3}(\partial_s^j k) \partial_s^j k ds + \int_{\mathbb{S}_t} |\partial_s^j k|^2 (\lambda_s - k^2) ds \\
&= -2 \int_{\mathbb{S}_t} |\partial_s^{j+1} k|^2 ds + \int_{\mathbb{S}_t} \partial_s (\lambda |\partial_s^j k|^2) ds + \int_{\mathbb{S}_t} \mathfrak{p}_{2j+4}(\partial_s^j k) ds \\
&\quad - 2 \sum_{p=1}^m \sum_{i=1}^3 \partial_s^j k^{pi} \partial_s^{j+1} k^{pi} \Big|_{\text{at the 3-point } O^p} + 2 \sum_{r=1}^l \partial_s^j k \partial_s^{j+1} k \Big|_{\text{at the end-point } P^r} \\
&\leq -2 \int_{\mathbb{S}_t} |\partial_s^{j+1} k|^2 ds + \int_{\mathbb{S}_t} \mathfrak{p}_{2j+4}(\partial_s^j k) ds + l C_j C_{j+1} \\
&\quad - \sum_{p=1}^m \sum_{i=1}^3 2 \partial_s^j k^{pi} \partial_s^{j+1} k^{pi} + \lambda^{pi} |\partial_s^j k^{pi}|^2 \Big|_{\text{at the 3-point } O^p}
\end{aligned} \tag{5.3}$$

where we integrated by parts the first term on the second line and we estimated the contributions given by the end-points  $P^r$  by means of assumption (5.1).

In the case that we consider the end-points  $P^1, P^2, \dots, P^l$  to be fixed, we can assume that the terms  $C_j C_{j+1}$  are all zero in the above conclusion, by the following lemma.

**Lemma 5.6.** *If the end-points  $P^r$  of the network are fixed, then there holds  $\partial_s^j k = \partial_s^j \lambda = 0$ , for every even  $j \in \mathbb{N}$ .*

*Proof.* The first case  $j = 0$  simply follows from the fact that the velocity  $\underline{v} = \lambda \tau + k \nu$  is always zero at the fixed end-points  $P^r$ .

We argue by induction, we suppose that for every even natural  $l \leq j - 2$  we have  $\partial_s^l k = \partial_s^l \lambda = 0$ , then, by using the first equation in Lemma 5.3, we get

$$\partial_s^j k = \partial_t \partial_s^{j-2} k - \lambda \partial_s^{j-1} k - \mathfrak{p}_{j+1}(\partial_s^{j-2} k)$$

at every end-point  $P^r$ .

We already know that  $\lambda = 0$  and by the inductive hypothesis  $\partial_s^{j-2} k = 0$ , thus  $\partial_t \partial_s^{j-2} k = 0$ . Since  $\mathfrak{p}_{j+1}(\partial_s^{j-2} k)$  is a sum of terms like  $C \prod_{l=0}^{j-2} (\partial_s^l k)^{\alpha_l}$  with  $\sum_{l=0}^{j-2} (l+1)\alpha_l = j+1$  which is odd, at least one of the terms of this sum has to be odd, hence at least for one index  $l$ , the product  $(l+1)\alpha_l$  is odd. It follows that at least for one even  $l$  the exponent  $\alpha_l$  is nonzero. Hence, at least one even derivatives is present in every monomial of  $\mathfrak{p}_{j+1}(\partial_s^{j-2} k)$ , which contains only derivatives up to the order  $(j-2)$ . Again, by the inductive hypothesis we then conclude that at the end-points  $\partial_s^j k = 0$ .

We can deal with  $\lambda$  similarly, by means of the relations in Lemma 5.3.  $\square$

In the very special case  $j = 0$  we get explicitly

$$\frac{d}{dt} \int_{\mathbb{S}_t} k^2 ds \leq -2 \int_{\mathbb{S}_t} |k_s|^2 ds + \int_{\mathbb{S}_t} k^4 ds - \sum_{p=1}^m \sum_{i=1}^3 2 k^{pi} k_s^{pi} + \lambda^{pi} |k^{pi}|^2 \Big|_{\text{at the 3-point } O^p} + l C_0 C_1$$

where the two constants  $C_0$  and  $C_1$  come from assumption (5.1).

Then, recalling relation (3.5), we have  $\sum_{i=1}^3 k^{pi} k_s^{pi} + \lambda^{pi} |k^{pi}|^2 \Big|_{\text{at the 3-point } O^p} = 0$ , and substituting above,

$$\frac{d}{dt} \int_{\mathbb{S}_t} k^2 ds \leq -2 \int_{\mathbb{S}_t} |k_s|^2 ds + \int_{\mathbb{S}_t} k^4 ds + \sum_{p=1}^m \sum_{i=1}^3 \lambda^{pi} |k^{pi}|^2 \Big|_{\text{at the 3-point } O^p} + l C_0 C_1, \tag{5.4}$$

hence, we lowered the maximum order of the space derivatives of the curvature in the 3-point terms, particular now it is lower than the one of the “nice” negative integral.

As we have just seen for the case  $j = 0$ , also for the general case we want to simplify the term  $\sum_{i=1}^3 2\partial_s^j k^{pi} \partial_s^{j+1} k^{pi} + \lambda^{pi} |\partial_s^j k^{pi}|^2 \Big|_{\text{at the 3-point } O^p}$  in order to control it.

Using formulas in Lemma 5.3, we have (see [63, pp. 258–259], for details)

$$\begin{aligned} & 2\partial_s^j k \partial_s^{j+1} k + \lambda |\partial_s^j k|^2 \\ &= 2\partial_t^{j/2} k \cdot \partial_t^{j/2} (k_s + k\lambda) + \mathbf{q}_{j+1}(\partial_t^{j/2-1} \lambda, \partial_s^{j-1} k) \cdot \partial_t^{j/2} k_s + \mathbf{q}_{2j+3}(\partial_t^{j/2} \lambda, \partial_s^j k). \end{aligned}$$

We now examine the term  $\mathbf{q}_{j+1}(\partial_t^{j/2-1} \lambda, \partial_s^{j-1} k) \cdot \partial_t^{j/2} k_s$ , which, by using Lemma 5.3, can be written as  $\partial_t \mathbf{q}_{2j+1}(\partial_t^{j/2-1} \lambda, \partial_s^{j-1} k) + \mathbf{q}_{2j+3}(\partial_t^{j/2} \lambda, \partial_s^j k)$  (see [63, pp. 258–259], for details). It follows that

$$\begin{aligned} & \sum_{p=1}^m \sum_{i=1}^3 2\partial_s^j k^{pi} \partial_s^{j+1} k^{pi} + \lambda^{pi} |\partial_s^j k^{pi}|^2 \Big|_{\text{at the 3-point } O^p} \\ &= \sum_{p=1}^m \sum_{i=1}^3 \partial_t \mathbf{q}_{2j+1}(\partial_t^{j/2-1} \lambda^{pi}, \partial_s^{j-1} k^{pi}) + \mathbf{q}_{2j+3}(\partial_t^{j/2} \lambda^{pi}, \partial_s^j k^{pi}) \Big|_{\text{at the 3-point } O^p} \end{aligned}$$

Resuming, if  $j \geq 2$  is even, we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{S}_t} |\partial_s^j k|^2 ds &\leq -2 \int_{\mathbb{S}_t} |\partial_s^{j+1} k|^2 ds + \int_{\mathbb{S}_t} \mathbf{p}_{2j+4}(\partial_s^j k) ds + lC_j C_{j+1} \\ &\quad + \sum_{p=1}^m \sum_{i=1}^3 \partial_t \mathbf{q}_{2j+1}(\partial_t^{j/2-1} \lambda^{pi}, \partial_s^{j-1} k^{pi}) + \mathbf{q}_{2j+3}(\partial_t^{j/2} \lambda^{pi}, \partial_s^j k^{pi}) \Big|_{\text{at the 3-point } O^p}. \end{aligned}$$

Now, the key tool to estimate the terms  $\int_{\mathbb{S}_t} \mathbf{p}_{2j+4}(\partial_s^j k) ds$  and  $\sum_{i=1}^3 \mathbf{q}_{2j+3}(\partial_t^{j/2} \lambda^{pi}, \partial_s^j k^{pi}) \Big|_{\text{at the 3-point } O^p}$  are the following Gagliardo–Nirenberg interpolation inequalities (see [69, Section 3, pp. 257–263]).

**Proposition 5.7.** *Let  $\gamma$  be a  $C^\infty$ , regular curve in  $\mathbb{R}^2$  with finite length  $L$ . If  $u$  is a  $C^\infty$  function defined on  $\gamma$  and  $m \geq 1, p \in [2, +\infty]$ , we have the estimates*

$$\|\partial_s^n u\|_{L^p} \leq C_{n,m,p} \|\partial_s^m u\|_{L^2}^\sigma \|u\|_{L^2}^{1-\sigma} + \frac{B_{n,m,p}}{L^{m\sigma}} \|u\|_{L^2} \quad (5.5)$$

for every  $n \in \{0, \dots, m-1\}$  where

$$\sigma = \frac{n+1/2-1/p}{m}$$

and the constants  $C_{n,m,p}$  and  $B_{n,m,p}$  are independent of  $\gamma$ . In particular, if  $p = +\infty$ ,

$$\|\partial_s^n u\|_{L^\infty} \leq C_{n,m} \|\partial_s^m u\|_{L^2}^\sigma \|u\|_{L^2}^{1-\sigma} + \frac{B_{n,m}}{L^{m\sigma}} \|u\|_{L^2} \quad \text{with} \quad \sigma = \frac{n+1/2}{m}. \quad (5.6)$$

After estimating the integral of every monomial of  $\mathbf{p}_{2j+4}(\partial_s^j k)$  by mean of the Hölder inequality, one uses the Gagliardo–Nirenberg estimates on the result, concluding that

$$\int_{\mathbb{S}_t} \mathbf{p}_{2j+4}(\partial_s^j k) ds \leq 1/4 \int_{\mathbb{S}_t} |\partial_s^{j+1} k|^2 ds + C \left( \int_{\mathbb{S}_t} k^2 ds \right)^{2j+3} + C,$$

where the constant  $C$  depends only on  $j \in \mathbb{N}$  and the lengths of the curves of the network (see [63, pp. 260–262], for details).

Any term  $\sum_{i=1}^3 \mathbf{q}_{2j+3}(\partial_t^{j/2} \lambda^{pi}, \partial_s^j k^{pi}) \Big|_{\text{at the 3-point } O^p}$  can be estimated similarly.

Hence, for every even  $j \geq 2$  we can finally write

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{S}_t} |\partial_s^j k|^2 ds &\leq - \int_{\mathbb{S}_t} |\partial_s^{j+1} k|^2 ds + C \left( \int_{\mathbb{S}_t} k^2 ds \right)^{2j+3} + C + lC_j C_{j+1} \\ &\quad + \partial_t \sum_{p=1}^m \sum_{i=1}^3 \mathbf{q}_{2j+1}(\partial_t^{j/2-1} \lambda^{pi}, \partial_s^{j-1} k^{pi}) \Big|_{\text{at the 3-point } O^p} \\ &\leq C \left( \int_{\mathbb{S}_t} k^2 ds \right)^{2j+3} + \partial_t \sum_{p=1}^m \sum_{i=1}^3 \mathbf{q}_{2j+1}(\partial_t^{j/2-1} \lambda^{pi}, \partial_s^{j-1} k^{pi}) \Big|_{\text{at the 3-point } O^p} + C + lC_j C_{j+1}. \end{aligned} \quad (5.7)$$

Recalling the computation in the special case  $j = 0$ , this argument gives the same final estimate without the contributions coming from the 3-points:

$$\left| \frac{d}{dt} \int_{\mathbb{S}_t} k^2 ds \right| \leq C \left( \int_{\mathbb{S}_t} k^2 ds \right)^3 + C + lC_0C_1. \quad (5.8)$$

Integrating (5.7) in time on  $[0, t]$  and estimating we get

$$\begin{aligned} \int_{\mathbb{S}_t} |\partial_s^j k|^2 ds &\leq \int_{\mathbb{S}_0} |\partial_s^j k|^2 ds + C \int_0^t \left( \int_{\mathbb{S}_\xi} k^2 ds \right)^{2j+3} d\xi + Ct + lC_j C_{j+1} t \\ &\quad + \sum_{p=1}^m \sum_{i=1}^3 \mathfrak{q}_{2j+1}(\partial_t^{j/2-1} \lambda^{p_i}(0, t), \partial_s^{j-1} k^{p_i}(0, t)) \\ &\quad - \mathfrak{q}_{2j+1}(\partial_t^{j/2-1} \lambda^{p_i}(0, 0), \partial_s^{j-1} k^{p_i}(0, 0)) \\ &\leq C \int_0^t \left( \int_{\mathbb{S}_\xi} k^2 ds \right)^{2j+3} d\xi + \|\mathfrak{p}_{2j+1}(|\partial_s^{j-1} k|)\|_{L^\infty} + Ct + lC_j C_{j+1} t + C, \end{aligned}$$

where in the last passage we used Remark 5.4. The constant  $C$  depends only on  $j \in \mathbb{N}$  and on the network  $\mathbb{S}_0$ .

Interpolating again by means of inequalities (5.6), one gets

$$\|\mathfrak{p}_{2j+1}(|\partial_s^{j-1} k|)\|_{L^\infty} \leq 1/2 \|\partial_s^j k\|_{L^2}^2 + C \|k\|_{L^2}^{4j+2}.$$

Hence, putting all together, for every even  $j \in \mathbb{N}$ , we conclude

$$\int_{\mathbb{S}_t} |\partial_s^j k|^2 ds \leq C \int_0^t \left( \int_{\mathbb{S}_\xi} k^2 ds \right)^{2j+3} d\xi + C \left( \int_{\mathbb{S}_t} k^2 ds \right)^{2j+1} + Ct + lC_j C_{j+1} t + C.$$

Passing from integral to  $L^\infty$  estimates, by using inequalities (5.6), we have the following proposition.

**Proposition 5.8.** *If assumption (5.1) holds, the lengths of all the curves are uniformly positively bounded from below and the  $L^2$  norm of  $k$  is uniformly bounded on  $[0, T]$ , then the curvature of  $\mathbb{S}_t$  and all its space derivatives are uniformly bounded in the same time interval by some constants depending only on the  $L^2$  integrals of the space derivatives of  $k$  on the initial network  $\mathbb{S}_0$ .*

By using the relations in Lemma 5.3, one then gets also estimates for every time and space derivatives of  $\lambda$  which finally imply estimates on all the derivatives of the maps  $\gamma^i$ , stated in the next proposition (see [63, pp. 263–266] for details).

**Proposition 5.9.** *If  $\mathbb{S}_t$  is a  $C^\infty$  special evolution of the initial network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i$ , satisfying assumption (5.1), such that the lengths of the  $n$  curves are uniformly bounded away from zero and the  $L^2$  norm of the curvature is uniformly bounded by some constants in the time interval  $[0, T]$ , then*

- *all the derivatives in space and time of  $k$  and  $\lambda$  are uniformly bounded in  $[0, 1] \times [0, T]$ ,*
- *all the derivatives in space and time of the curves  $\gamma^i(x, t)$  are uniformly bounded in  $[0, 1] \times [0, T]$ ,*
- *the quantities  $|\gamma_x^i(x, t)|$  are uniformly bounded from above and away from zero in  $[0, 1] \times [0, T]$ .*

*All the bounds depend only on the uniform controls on the  $L^2$  norm of  $k$ , on the lengths of the curves of the network from below, on the constants  $C_j$  in assumption (5.1), on the  $L^\infty$  norms of the derivatives of the curves  $\sigma^i$  and on the bound from above and below on  $|\sigma_x^i(x, t)|$ , for the curves describing the initial network  $\mathbb{S}_0$ .*

Now, we work out a second set of estimates where everything is controlled – still under the assumption (5.1) – only by the  $L^2$  norm of the curvature and the inverses of the lengths of the curves at time zero.

As before we consider the  $C^\infty$  special curvature flow  $\mathbb{S}_t$  of a smooth network  $\mathbb{S}_0$  in the time interval  $[0, T]$ , composed by  $n$  curves  $\gamma^i(\cdot, t) : [0, 1] \rightarrow \overline{\Omega}$  with  $m$  3-points  $O^1, O^2, \dots, O^m$  and  $l$  end-points  $P^1, P^2, \dots, P^l$ , satisfying assumption (5.1).



**Proposition 5.10.** *For every  $M > 0$  there exists a time  $T_M \in (0, T)$ , depending only on the structure of the network and on the constants  $C_0$  and  $C_1$  in assumption (5.1), such that if the square of the  $L^2$  norm of the curvature and the inverses of the lengths of the curves of  $\mathbb{S}_0$  are bounded by  $M$ , then the square of the  $L^2$  norm of  $k$  and the inverses of the lengths of the curves of  $\mathbb{S}_t$  are smaller than  $2(n+1)M+1$ , for every time  $t \in [0, T_M]$ .*

*Proof.* The evolution equations for the lengths of the  $n$  curves are given by

$$\frac{dL^i(t)}{dt} = \lambda^i(1, t) - \lambda^i(0, t) - \int_{\gamma^i(\cdot, t)} k^2 ds,$$

then, recalling computation (5.4), we have

$$\begin{aligned} \frac{d}{dt} \left( \int_{\mathbb{S}_t} k^2 ds + \sum_{i=1}^n \frac{1}{L^i} \right) &\leq -2 \int_{\mathbb{S}_t} k_s^2 ds + \int_{\mathbb{S}_t} k^4 ds + 6m \|k\|_{L^\infty}^3 + lC_0 C_1 - \sum_{i=1}^n \frac{1}{(L^i)^2} \frac{dL^i}{dt} \\ &= -2 \int_{\mathbb{S}_t} k_s^2 ds + \int_{\mathbb{S}_t} k^4 ds + 6m \|k\|_{L^\infty}^3 + lC_0 C_1 \\ &\quad - \sum_{i=1}^n \frac{\lambda^i(1, 0) - \lambda^i(0, t) + \int_{\gamma^i(\cdot, t)} k^2 ds}{(L^i)^2} \\ &\leq -2 \int_{\mathbb{S}_t} k_s^2 ds + \int_{\mathbb{S}_t} k^4 ds + 6m \|k\|_{L^\infty}^3 + lC_0 C_1 \\ &\quad + 2 \sum_{i=1}^n \frac{\|k\|_{L^\infty} + C_0}{(L^i)^2} + \sum_{i=1}^n \frac{\int_{\mathbb{S}_t} k^2 ds}{(L^i)^2} \\ &\leq -2 \int_{\mathbb{S}_t} k_s^2 ds + \int_{\mathbb{S}_t} k^4 ds + (6m + 2n/3) \|k\|_{L^\infty}^3 + lC_0 C_1 + 2nC_0^3/3 \\ &\quad + \frac{n}{3} \left( \int_{\mathbb{S}_t} k^2 ds \right)^3 + \frac{2}{3} \sum_{i=1}^n \frac{1}{(L^i)^3} \end{aligned}$$

where we used Young inequality in the last passage.

Interpolating as before (and applying again Young inequality) but keeping now in evidence the terms depending on  $L^i$  in inequalities (5.5), we obtain

$$\begin{aligned} \frac{d}{dt} \left( \int_{\mathbb{S}_t} k^2 ds + \sum_{i=1}^n \frac{1}{L^i} \right) &\leq - \int_{\mathbb{S}_t} k_s^2 ds + C \left( \int_{\mathbb{S}_t} k^2 ds \right)^3 + C \sum_{i=1}^n \frac{\left( \int_{\mathbb{S}_t} k^2 ds \right)^2}{L^i} \\ &\quad + C \sum_{i=1}^n \frac{\left( \int_{\mathbb{S}_t} k^2 ds \right)^{3/2}}{(L^i)^{3/2}} + C \sum_{i=1}^n \frac{1}{(L^i)^3} + C \\ &\leq C \left( \int_{\mathbb{S}_t} k^2 ds \right)^3 + C \sum_{i=1}^n \frac{1}{(L^i)^3} + C \\ &\leq C \left( \int_{\mathbb{S}_t} k^2 ds + \sum_{i=1}^n \frac{1}{L^i} + 1 \right)^3, \end{aligned}$$

with a constant  $C$  depending only on the structure of the network and on the constants  $C_0$  and  $C_1$  in assumption (5.1).

This means that the positive function  $f(t) = \int_{\mathbb{S}_t} k^2 ds + \sum_{i=1}^n \frac{1}{L^i(t)} + 1$  satisfies the differential inequality  $f' \leq C f^3$ , hence, after integration

$$f^2(t) \leq \frac{f^2(0)}{1 - 2Ct f^2(0)} \leq \frac{f^2(0)}{1 - 2Ct[(n+1)M+1]}$$

then, if  $t \leq T_M = \frac{3}{8C[(n+1)M+1]}$ , we get  $f(t) \leq 2f(0)$ . Hence,

$$\int_{\mathbb{S}_t} k^2 ds + \sum_{i=1}^n \frac{1}{L^i(t)} \leq 2 \int_{\mathbb{S}_0} k^2 ds + 2 \sum_{i=1}^n \frac{1}{L^i(0)} + 1 \leq 2[(n+1)M] + 1.$$

□

By means of this proposition we can strengthen the conclusion of Proposition 5.9.

**Corollary 5.11.** *In the hypothesis of the previous proposition, in the time interval  $[0, T_M]$  all the bounds in Proposition 5.9 depend only on the  $L^2$  norm of  $k$  on  $\mathbb{S}_0$ , on the constants  $C_j$  in assumption (5.1), on the  $L^\infty$  norms of the derivatives of the curves  $\sigma^i$ , on the bound from above and below on  $|\sigma_x^i(x, t)|$  and on the lengths of the curves of the initial network  $\mathbb{S}_0$ .*

From now on we assume that the  $L^2$  norm of the curvature and the inverses of the lengths of the curves are bounded in the interval  $[0, T_M]$ .

Considering  $j \in \mathbb{N}$  even, if we differentiate the function

$$\int_{\mathbb{S}_t} k^2 + tk_s^2 + \frac{t^2 k_{ss}^2}{2!} + \cdots + \frac{t^j |\partial_s^j k|^2}{j!} ds,$$

and we estimate with interpolation inequalities as before (see [63, pp. 268–269], for details), we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{S}_t} k^2 + tk_s^2 + \frac{t^2 k_{ss}^2}{2!} + \cdots + \frac{t^j |\partial_s^j k|^2}{j!} ds \\ & \leq -\varepsilon \int_{\mathbb{S}_t} k_s^2 + tk_{ss}^2 + t^2 k_{sss}^2 + \cdots + t^j |\partial_s^{j+1} k|^2 ds + C \\ & \quad + \partial_t \sum_{p=1}^m \sum_{i=1}^3 t^2 \mathbf{q}_5(\lambda^{pi}, k_s^{pi}) + t^4 \mathbf{q}_9(\partial_t \lambda^{pi}, k_{sss}^{pi}) + \cdots + t^j \mathbf{q}_{2j+1}(\partial_t^{j/2-1} \lambda^{pi}, \partial_s^{j-1} k^{pi}) \Big|_{\text{at the 3-point } O^p} \\ & \quad + C \sum_{p=1}^m \sum_{i=1}^3 tk_s^{pi} k_{ss}^{pi} + t^3 k_{sss}^{pi} k_{ssss}^{pi} + \cdots + t^{j-1} \partial_s^{j-1} k^{pi} \partial_s^j k^{pi} \Big|_{\text{at the 3-point } O^p} \end{aligned} \quad (5.9)$$

in the time interval  $[0, T_M]$ , where  $\varepsilon > 0$  and  $C$  are two constants depending only on the  $L^2$  norm of the curvature, the constants in assumption (5.1) and the inverses of the lengths of the  $n$  curves of  $\mathbb{S}_0$ .

We proceed as we did before for the computation of  $\frac{d}{dt} \int_{\mathbb{S}_t} |\partial_s^j k|^2 ds$ .

First we deal with the last line,

$$\sum_{i=1}^3 tk_s^{pi} k_{ss}^{pi} + t^3 k_{sss}^{pi} k_{ssss}^{pi} + \cdots + t^{j-1} \partial_s^{j-1} k^{pi} \partial_s^j k^{pi} \Big|_{\text{at the 3-point}}.$$

By formulas in Lemma 5.3 and by Remark 5.4, we can write, for any term  $\sum_{i=1}^3 t^{h-1} \partial_s^{h-1} k^i \partial_s^h k^i \Big|_{\text{at the 3-point}}$ ,

$$\begin{aligned} \sum_{i=1}^3 t^{h-1} \partial_s^{h-1} k^i \partial_s^h k^i \Big|_{\text{at the 3-point}} &= \sum_{i=1}^3 t^{h-1} \mathbf{q}_{2h+1}(\partial_t^{h/2-1} \lambda^i, \partial_s^{h-1} k^i) \\ &\quad + t^{h-1} \partial_s^h k^i \cdot \mathbf{q}_h(\partial_t^{h/2-1} \lambda^i, \partial_s^{h-2} k^i) \Big|_{\text{at the 3-point}} \\ &\leq t^{h-1} \|\mathbf{p}_{2h+1}(|\partial_s^{h-1} k|)\|_{L^\infty} + t^{h-1} \|\partial_s^h k\|_{L^\infty} \|\mathbf{p}_h(|\partial_s^{h-2} k|)\|_{L^\infty} \end{aligned}$$

(see [63, p. 270], for details).

The term  $t^{h-1} \|\mathbf{p}_{2h+1}(|\partial_s^{h-1} k|)\|_{L^\infty}$  is controlled as before by a small fraction of the term  $t^{h-1} \int_{\mathbb{S}_t} |\partial_s^h k|^2 ds$  and a possibly large multiple of  $t^{h-1}$  times some power of the  $L^2$  norm of  $k$  (which is bounded), whereas  $t^{h-1} \|\partial_s^h k\|_{L^\infty} \|\mathbf{p}_h(|\partial_s^{h-2} k|)\|_{L^\infty}$  is the critical term.

Again by means of interpolation inequalities (5.6) one estimates  $\|\partial_s^h k\|_{L^\infty}$ ,  $\|\mathbf{p}_h(\partial_s^{h-2} k)\|_{L^\infty}$  and  $\|\partial_s^h k\|_{L^2}$  with the  $L^2$  norm of  $k$  and its derivatives. After some computation (see [63, pp. 270–271], for details), one gets

$$\sum_{i=1}^3 t^{h-1} \partial_s^{h-1} k^i \partial_s^h k^i \Big|_{\text{at the 3-point}} \leq \varepsilon_h/2 \left( t^h \int_{\mathbb{S}_t} |\partial_s^{h+1} k|^2 ds + t^{h-1} \int_{\mathbb{S}_t} |\partial_s^h k|^2 ds + Ct^h \right) + C/t^{\theta_h}$$

with  $\theta_h < 1$  and some small  $\varepsilon_h > 0$ .

We apply this argument for every even  $h$  from 2 to  $j$ , choosing accurately small values  $\varepsilon_j$ . Hence, we can continue estimate (5.9) as follows,

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{S}_t} k^2 + tk_s^2 + \frac{t^2 k_{ss}^2}{2!} + \cdots + \frac{t^j |\partial_s^j k|^2}{j!} ds \\
& \leq -\varepsilon/2 \int_{\mathbb{S}_t} k^2 + tk_{ss}^2 + t^2 k_{ssss}^2 + \cdots + t^j |\partial_s^{j+1} k|^2 ds + C + C/t^{\theta_2} + \cdots + C/t^{\theta_j} \\
& \quad + \partial_t \sum_{i=1}^3 t^2 q_5(\lambda^i, k_s^i) + t^4 q_9(\partial_t \lambda^i, k_{ss}^i) + \cdots + t^j q_{2j+1}(\partial_t^{j/2-1} \lambda^i, \partial_s^{j-1} k^i) \Big|_{\text{at the 3-point}} \\
& \leq C + C/t^\theta + \partial_t \sum_{i=1}^3 t^2 q_5(\lambda^i, k_s^i) + t^4 q_9(\partial_t \lambda^i, k_{ss}^i) + \cdots + t^j q_{2j+1}(\partial_t^{j/2-1} \lambda^i, \partial_s^{j-1} k^i) \Big|_{\text{at the 3-point}}
\end{aligned}$$

for some  $\theta < 1$ .

Integrating this inequality in time on  $[0, t]$  with  $t \leq T_M$  and taking into account Remark 5.4, we get

$$\begin{aligned}
& \int_{\mathbb{S}_t} k^2 + tk_s^2 + \frac{t^2 k_{ss}^2}{2!} + \cdots + \frac{t^j |\partial_s^j k|^2}{j!} ds \\
& \leq \int_{\mathbb{S}_0} k^2 ds + CT_M + CT_M^{(1-\theta)} \\
& \quad + \sum_{i=1}^3 t^2 q_5(\lambda^i, k_s^i) + t^4 q_9(\partial_t \lambda^i, k_{ss}^i) + \cdots + t^j q_{2j+1}(\partial_t^{j/2-1} \lambda^i, \partial_s^{j-1} k^i) \Big|_{\text{at the 3-point}} \\
& \leq \int_{\mathbb{S}_0} k^2 ds + C + t^2 \|p_5(|k_s|)\|_{L^\infty} + t^4 \|p_9(|k_{ss}|)\|_{L^\infty} + \cdots + t^j \|p_{2j+1}(|\partial_s^{j-1} k|)\|_{L^\infty}.
\end{aligned}$$

Now we absorb all the polynomial terms, after interpolating each one of them between the corresponding “good” integral in the left member and some power of the  $L^2$  norm of  $k$ , as we did in showing Proposition 5.8, hence we finally obtain for every even  $j \in \mathbb{N}$ ,

$$\int_{\mathbb{S}_t} k^2 + tk_s^2 + \frac{t^2 k_{ss}^2}{2!} + \cdots + \frac{t^j |\partial_s^j k|^2}{j!} ds \leq \overline{C}_j$$

with  $t \in [0, T_M]$  and a constant  $\overline{C}_j$  depending only on the constants in assumption (5.1) and the bounds on  $\int_{\mathbb{S}_0} k^2 ds$  and on the inverses of the lengths of the curves of the initial network  $\mathbb{S}_0$ .

This family of inequalities clearly implies

$$\int_{\mathbb{S}_t} |\partial_s^j k|^2 ds \leq \frac{C_j j!}{t^j} \quad \text{for every even } j \in \mathbb{N}.$$

Then, passing as before from integral to  $L^\infty$  estimates by means of inequalities (5.6), we have the following proposition.

**Proposition 5.12.** *For every  $\mu > 0$  the curvature and all its space derivatives of  $\mathbb{S}_t$  are uniformly bounded in the time interval  $[\mu, T_M]$  (where  $T_M$  is given by Proposition 5.10) by some constants depending only on  $\mu$ , the constants in assumption (5.1) and the bounds on  $\int_{\mathbb{S}_0} k^2 ds$  and on the inverses of the lengths of the curves of the initial network  $\mathbb{S}_0$ .*

By means of these a priori estimates we can now work out some results about the smooth flow of an initial regular geometrically smooth network  $\mathbb{S}_0$ . Notice that these are examples of how to use the previous estimates on special smooth flows in order to get conclusion on general flows or even only  $C^\infty$  flows, as we mentioned at the beginning of this section.

**Theorem 5.13.** *If  $[0, T)$ , with  $T < +\infty$ , is the maximal time interval of existence of a  $C^\infty$  curvature flow of an initial geometrically smooth network  $\mathbb{S}_0$ , then*

1. *either the inferior limit of the length of at least one curve of  $\mathbb{S}_t$  is zero, as  $t \rightarrow T$ ,*

2. or  $\overline{\lim}_{t \rightarrow T} \int_{\mathbb{S}_t} k^2 ds = +\infty$ .

Moreover, if the lengths of the  $n$  curves are uniformly positively bounded from below, then this superior limit is actually a limit and there exists a positive constant  $C$  such that

$$\int_{\mathbb{S}_t} k^2 ds \geq \frac{C}{\sqrt{T-t}},$$

for every  $t \in [0, T)$ .

*Proof.* We can  $C^\infty$  reparametrize the flow  $\mathbb{S}_t$  in order that it becomes a special smooth flow  $\tilde{\mathbb{S}}_t$  in  $[0, T)$ . If the lengths of the curves of  $\mathbb{S}_t$  are uniformly bounded away from zero and the  $L^2$  norm of  $k$  is bounded, the same holds for the networks  $\tilde{\mathbb{S}}_t$ , then, by Proposition 5.9 and Ascoli–Arzelà Theorem, the network  $\tilde{\mathbb{S}}_t$  converges in  $C^\infty$  to a smooth network  $\tilde{\mathbb{S}}_T$  as  $t \rightarrow T$ . Then, applying Theorem 4.18 to  $\tilde{\mathbb{S}}_T$  we could restart the flow obtaining a  $C^\infty$  special curvature flow in a longer time interval. Reparametrizing back this last flow, we get a  $C^\infty$  “extension” in time of the flow  $\mathbb{S}_t$ , hence contradicting the maximality of the interval  $[0, T)$ .

Now, considering again the flow  $\tilde{\mathbb{S}}_t$ , by means of differential inequality (5.8), we have

$$\frac{d}{dt} \int_{\tilde{\mathbb{S}}_t} \tilde{k}^2 ds \leq C \left( \int_{\tilde{\mathbb{S}}_t} \tilde{k}^2 ds \right)^3 + C \leq C \left( 1 + \int_{\tilde{\mathbb{S}}_t} \tilde{k}^2 ds \right)^3,$$

which, after integration between  $t, r \in [0, T)$  with  $t < r$ , gives

$$\frac{1}{\left( 1 + \int_{\tilde{\mathbb{S}}_t} \tilde{k}^2 ds \right)^2} - \frac{1}{\left( 1 + \int_{\tilde{\mathbb{S}}_r} \tilde{k}^2 ds \right)^2} \leq C(r-t).$$

Then, if case (1) does not hold, we can choose a sequence of times  $r_j \rightarrow T$  such that  $\int_{\tilde{\mathbb{S}}_{r_j}} \tilde{k}^2 ds \rightarrow +\infty$ . Putting  $r = r_j$  in the inequality above and passing to the limit, as  $j \rightarrow \infty$ , we get

$$\frac{1}{\left( 1 + \int_{\tilde{\mathbb{S}}_t} \tilde{k}^2 ds \right)^2} \leq C(T-t),$$

hence, for every  $t \in [0, T)$ ,

$$\int_{\tilde{\mathbb{S}}_t} \tilde{k}^2 ds \geq \frac{C}{\sqrt{T-t}} - 1 \geq \frac{C}{\sqrt{T-t}},$$

for some positive constant  $C$  and  $\lim_{t \rightarrow T} \int_{\tilde{\mathbb{S}}_t} k^2 ds = +\infty$ .

By the invariance of the curvature by reparametrization, this last estimate implies the same estimate for the flow  $\mathbb{S}_t$ .  $\square$

This theorem obviously implies the following corollary.

**Corollary 5.14.** *If  $[0, T)$ , with  $T < +\infty$ , is the maximal time interval of existence of a  $C^\infty$  curvature flow of an initial geometrically smooth network  $\mathbb{S}_0$  and the lengths of the curves are uniformly bounded away from zero, then*

$$\max_{\mathbb{S}_t} k^2 \geq \frac{C}{\sqrt{T-t}} \rightarrow +\infty, \quad (5.10)$$

as  $t \rightarrow T$ .

*Remark 5.15.* In the case of the evolution  $\gamma_t$  of a single closed curve in the plane there exists a constant  $C > 0$  such that if at time  $T > 0$  a singularity develops, then

$$\max_{\gamma_t} k^2 \geq \frac{C}{T-t}$$

for every  $t \in [0, T)$  (see [42]).

If this lower bound on the rate of blowing up of the curvature (which is clearly stronger than the one in inequality (5.10)) holds also in the case of the evolution of a network is an open problem (even if the network is a triod).

We conclude this section with the following estimate from below on the maximal time of smooth existence.

**Proposition 5.16.** *For every  $M > 0$  there exists a positive time  $T_M$  such that if the  $L^2$  norm of the curvature and the inverses of the lengths of the geometrically smooth network  $\mathbb{S}_0$  are bounded by  $M$ , then the maximal time of existence  $T > 0$  of a  $C^\infty$  curvature flow of  $\mathbb{S}_0$  is larger than  $T_M$ .*

*Proof.* As before, considering again the reparametrized special curvature flow  $\tilde{\mathbb{S}}_t$ , by Proposition 5.10 in the interval  $[0, \min\{T_M, T\})$  the  $L^2$  norm of  $\tilde{k}$  and the inverses of the lengths of the curves of  $\tilde{\mathbb{S}}_t$  are bounded by  $2M^2 + 6M$ .

Then, by Theorem 5.13, the value  $\min\{T_M, T\}$  cannot coincide with the maximal time of existence of  $\tilde{\mathbb{S}}_t$  (hence of  $\mathbb{S}_t$ ), so it must be  $T > T_M$ .  $\square$

## 6 Short time existence II

First we consider a  $C^\infty$  flow by curvature  $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i([0, 1], t)$  and we discuss what happens if we reparametrize every curve of the network proportionally to arclength.

If we consider smooth functions  $\varphi^i : [0, 1] \times [0, T) \rightarrow [0, 1]$  and the reparametrizations  $\tilde{\gamma}^i(x, t) = \gamma^i(\varphi^i(x, t), t)$ , imposing that  $|\tilde{\gamma}_x^i|$  is constant, we must have that  $|\gamma_x^i(\varphi^i(x, t), t)|\varphi_x^i(x, t) = L^i(t)$ , where  $L^i(t)$  is the length of the curve  $\gamma^i$  at time  $t$ .

It follows that  $\varphi^i(x, t)$  can be obtained by integrating the ODE

$$\varphi_x^i(x, t) = L^i(t)/|\gamma_x^i(\varphi^i(x, t), t)|$$

with initial data  $\varphi^i(0, t) = 0$  and that it is  $C^\infty$ , as  $L^i$  and  $\gamma^i$  are  $C^\infty$ .

Being a reparametrization,  $\tilde{\gamma}^i$  is still a  $C^\infty$  curvature flow, that is,  $\tilde{\gamma}_t^i = \tilde{k}^i \tilde{\nu}^i + \tilde{\lambda}^i \tilde{\tau}^i$ . We want to determine the functions  $\tilde{\lambda}^i = \langle \tilde{\gamma}_t^i | \tilde{\tau}^i \rangle$ , differentiating this equation in arclength and keeping into account that  $\tilde{\gamma}_x(x, t) = L^i(t) \tilde{\tau}^i(x, t)$ , we get

$$\tilde{\lambda}_s^i = \frac{\langle \tilde{\gamma}_{tx}^i | \tilde{\tau}^i \rangle}{|\tilde{\gamma}_x^i|} + \langle \tilde{\gamma}_t^i | \partial_s \tilde{\tau}^i \rangle = \frac{\langle \partial_t(L^i \tilde{\tau}^i) | \tilde{\tau}^i \rangle}{L^i} + \langle \tilde{k}^i \tilde{\nu}^i + \tilde{\lambda}^i \tilde{\tau}^i | \tilde{k}^i \tilde{\nu}^i \rangle = \frac{\partial_t L^i}{L^i} + (\tilde{k}^i)^2.$$

This equation immediately says that  $\tilde{\lambda}_s^i - (\tilde{k}^i)^2$  is constant in space. Moreover, we know that  $\partial_t L^i(t) = \tilde{\lambda}^i(1, t) - \tilde{\lambda}^i(0, t) - \int_{\gamma^i(\cdot, t)} (\tilde{k}^i)^2 ds$  (see Proposition 5.1) and that the values of  $\tilde{\lambda}^i$  at the end-points or 3-points of the network are (uniformly) linearly related (hence, also bounded) to the values of  $\tilde{k}^i$ ; hence, we can conclude that  $\tilde{\lambda}_s^i$  is bounded by  $L^i(t)$  and a quadratic expression in  $\|\tilde{k}(\cdot, t)\|_{L^\infty}$ .

We show now that the geometrically unique solution starting from an initial  $C^{2+2\alpha}$  network which is geometrically 2-compatible (see Proposition 4.22) can be actually reparametrized to be a  $C^\infty$  curvature flow for every positive time (so that the geometric estimates of Section 5 can be applied). This clearly can be seen as a (geometric) parabolic regularization property.

**Theorem 6.1.** *For any initial, regular  $C^{2+2\alpha}$  network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$ , with  $\alpha \in (0, 1/2)$ , which is geometrically 2-compatible, the geometrically unique solution  $\gamma^i$  found in Proposition 4.22 can be reparametrized to be a  $C^\infty$  curvature flow on  $(0, T)$ , that is, the networks  $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i([0, 1], t)$  are geometrically smooth for every positive time.*

*Proof.* We first assume that  $\mathbb{S}_0$  is 2-compatible.

By inspecting the proof of Theorem 4.3 in [14] one can see that the solution to system (4.3) actually depends continuously in  $C^{2+2\alpha, 1+\alpha}$  on the initial data  $\sigma^i$  in the  $C^{2+\alpha}$  norm. Then, we approximate the network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$  in  $C^{2+2\alpha}$  with a family of smooth networks  $\mathbb{S}_j$  with the same end-points, composed of  $C^\infty$  curves  $\sigma_j^i \rightarrow \sigma^i$ , as  $j \rightarrow \infty$ . Hence, for every  $\varepsilon > 0$ , the smooth solutions of system (4.3) for these approximating initial networks, given by the curves  $\gamma_j^i(x, t) : [0, 1] \times [0, T - \varepsilon] \rightarrow \bar{\Omega}$ , converge as  $j \rightarrow \infty$  in  $C^{2+2\alpha, 1+\alpha}([0, 1] \times [0, T - \varepsilon])$  to the solution  $\gamma^i$  for the initial network  $\mathbb{S}_0$ . By the  $C^{2+2\alpha}$ -convergence, the inverses of the lengths of the initial curves, the integrals  $\int_{\mathbb{S}_j} k_j^2 ds$  and  $|\partial_x \sigma_j^i(x)|$  (from above and away from zero) for all the approximating networks are equibounded, thus Proposition 5.12

gives uniform estimates on the  $L^\infty$  norms of the curvature and of all its derivatives in every rectangle  $[0, 1] \times [\mu, T_M)$ , with  $\mu > 0$  and  $T_M \leq T$ .

We now reparametrize every curve  $\gamma_j^i(\cdot, t)$  and  $\gamma^i(\cdot, t)$  proportionally to arclength by some maps  $\varphi_j^i$  and  $\varphi^i$  as above. Notice that, since  $\gamma_j^i$  and  $\gamma^i$  are uniformly bounded in  $C^{2+2\alpha, 1+\alpha}$ , we have that the maps  $\partial_x \gamma_j^i$  and  $\partial_x \gamma^i$  uniformly bounded  $C^{1+2\alpha, 1/2+\alpha}$ . Hence, by a standard ODE's argument, the reparametrizing maps  $\varphi_j^i$  and  $\varphi^i$  above are also uniformly bounded in  $C^{1+2\alpha, 1/2+\alpha}$ , in particular, they are uniformly Hölder continuous in space and time. This means that the reparametrized maps  $\tilde{\gamma}_j^i$  converge uniformly to  $\tilde{\gamma}^i$  which is a (only continuous in  $t$ ) reparametrization of the original flow. It is easy to see that these latter gives a curvature flow of the arclength reparametrized network  $\tilde{\mathbb{S}}_0 = \bigcup_{i=1}^n (\sigma^i \circ \varphi^i(\cdot, 0)) [0, 1]$  which then still belongs to  $C^{2+2\alpha}$ .

As the curvature and all its arclength derivatives are invariant under reparametrization and the equibounded lengths of the curves also, the above uniform estimates hold also for the reparametrized maps  $\tilde{\gamma}_j^i$  in every rectangle  $[0, 1] \times [\mu, T_M)$ . Moreover, by the discussion about reparametrizing these curves proportional to arclength, it follows that we have uniform estimates also on  $\tilde{\lambda}_j^i$  and all their arclength derivatives for these flows in every rectangle  $[0, 1] \times [\mu, T_M)$ . Hence, the curves  $\tilde{\gamma}_j^i$ , possibly passing to a subsequence, actually converge in  $C^\infty([0, 1] \times [\mu, T_M))$ , for every  $\mu > 0$ , to the limit flow  $\tilde{\gamma}^i$  which then belongs to  $C^\infty([0, 1] \times (0, T)) \cap C^0([0, 1] \times [0, T))$ .

If  $\mathbb{S}_0$  were only geometrically 2-compatible, this procedure could have been done for the flow of its 2-compatible reparametrization, giving the same resulting flow, as the arclength reparametrized flow is the same for any two flows differing only for a reparametrization (the fact that the flow of a  $C^{2+2\alpha}$  geometrically 2-compatible initial network is a reparametrization of the flow of a 2-compatible  $C^{2+2\alpha}$  initial network is stated in Remark 4.24).

The last step is to find extensions  $\theta^i : [0, 1] \times [0, T) \rightarrow [0, 1]$  of the arclength reparametrizing maps  $\varphi^i(\cdot, 0) \in C^{2+2\alpha}$  which are in  $C^\infty([0, 1] \times (0, T))$  and satisfy  $\theta^i(x, 0) = \varphi^i(x, 0)$ ,  $\theta^i(0, t) = 0$ ,  $\theta^i(1, t) = 1$  and  $\theta_x^i(x, t) \neq 0$  for every  $x$  and  $t$ . This can be done, for instance, by means of time-dependent convolutions with smooth kernels. Then, the maps  $\tilde{\gamma}^i(\cdot, t) = \tilde{\gamma}^i([\theta^i(\cdot, t)]^{-1}, t)$  give a curvature flow of the network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$  which becomes immediately  $C^\infty$  for every positive time  $t > 0$ .  $\square$

As for every positive time, the flow obtained by this theorem is  $C^\infty$ , hence, every network  $\mathbb{S}_t$  is geometrically smooth, again by Remark 4.24 this flow can be reparametrized, from any positive time on, to be a  $C^\infty$  special smooth flow.

This argument can clearly be applied to any  $C^{2+2\alpha, 1+\alpha}$  curvature flow  $\mathbb{S}_t$  in a time interval  $(0, T)$ , being every network of this flow geometrically 2-compatible (Proposition 4.12), simply considering as initial network any  $\mathbb{S}_{t_0}$  with  $t_0 > 0$ .

**Corollary 6.2.** *Given any  $C^{2+2\alpha, 1+\alpha}$  curvature flow in an interval of time  $(0, T)$ , for every  $\mu > 0$ , the restricted flow  $\mathbb{S}_t$  for  $t \in [\mu, T)$  can be reparametrized to be a  $C^\infty$  special curvature flow in  $[\mu, T)$ . In particular, this applies to any  $C^{2+2\alpha, 1+\alpha}$  curvature flow of an initial, regular  $C^{2+2\alpha}$  geometrically 2-compatible network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$ .*

*Remark 6.3.* Even if this theorem and the corollary are sufficient for our purpose to study the singularity formation in the next sections, one would expect that by the usual standard parabolic regularization, the unique solution  $\gamma^i$  of system (4.3) for every initial  $C^{2+2\alpha}$  network, at least if it is 2-compatible, is actually  $C^\infty$  for every positive time, hence a special curvature flow. Another question is whether *any* curvature flow, hence only in  $C^{2,1}$ , can be reparametrized to be a  $C^\infty$  (special) curvature flow in  $[\mu, T)$ . These problem are actually open at the moment.

Also open is what is the largest class of initial networks admitting a special curvature flow.

**Open Problem 6.4.** The unique solution  $\gamma^i$  of system (4.3) for an initial  $C^{2+2\alpha}$  network  $\mathbb{S}_0$ , at least if it is 2-compatible, is  $C^\infty$  for every positive time?

**Open Problem 6.5.** Every curvature flow of a regular network can be reparametrized to be a  $C^\infty$  (special) curvature flow for every positive time?

**Open Problem 6.6.** What are the minimal regularity hypotheses on an initial network  $\mathbb{S}_0$  such that it admits a special curvature flow?



A consequence of these “geometric” parabolic results is the extension of Theorem 5.13 and Corollary 5.14 to any  $C^{2+\alpha,1+\alpha}$  curvature flow. As before, we apply such results to the reparametrized  $C^\infty$  special curvature flow given by Corollary 6.2, then it is clear that the conclusions holds also for the original flow since they are concerned only with the curvature and the lengths of the curves, which are invariant by reparametrization.

**Theorem 6.7.** *If  $T < +\infty$  is the maximal time interval of existence of a  $C^{2+\alpha,1+\alpha}$  curvature flow  $\mathbb{S}_t$ , then*

1. *either the inferior limit of the length of at least one curve of  $\mathbb{S}_t$  is zero, as  $t \rightarrow T$ ,*
2. *or  $\overline{\lim}_{t \rightarrow T} \int_{\mathbb{S}_t} k^2 ds = +\infty$ , hence, the curvature is not bounded as  $t \rightarrow T$ .*

*Moreover, if the lengths of the  $n$  curves are uniformly positively bounded from below, then this superior limit is actually a limit and there exists a positive constant  $C$  such that*

$$\int_{\mathbb{S}_t} k^2 ds \geq \frac{C}{\sqrt{T-t}} \quad \text{and} \quad \max_{\mathbb{S}_t} k^2 \geq \frac{C}{\sqrt{T-t}}$$

*for every  $t \in [0, T)$ .*

Thanks to Proposition 5.12, we can now improve Theorems 4.13 and 4.20 to show the existence of a curvature flow for a regular initial network  $\mathbb{S}_0$  which is only  $C^2$ .

**Theorem 6.8.** *For any initial  $C^2$  regular network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$  there exists a solution  $\gamma^i$  of Problem (2.3) in a maximal time interval  $[0, T)$ .*

*Such curvature flow  $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i([0, 1], t)$  is a smooth flow for every time  $t > 0$ , moreover, the unit tangents  $\tau^i$  are continuous in  $[0, 1] \times [0, T)$ , the functions  $k(\cdot, t)$  converge weakly in  $L^2(ds)$  to  $k(\cdot, 0)$ , as  $t \rightarrow 0$ , and the function  $\int_{\mathbb{S}_t} k^2 ds$  is continuous on  $[0, T)$ .*

*Proof.* We can approximate in  $W^{2,2}(0, 1)$  (hence in  $C^1([0, 1])$ ) the network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$  with a family of smooth networks  $\mathbb{S}_j$ , composed of  $C^\infty$  curves  $\sigma_j^i \rightarrow \sigma^i$ , as  $j \rightarrow \infty$  with the same end-points and satisfying  $\partial_x \sigma_j^i(0) = \partial_x \sigma^i(0)$ ,  $\partial_x \sigma_j^i(1) = \partial_x \sigma^i(1)$ .

By the convergence in  $W^{2,2}$  and in  $C^1$ , the inverses of the lengths of the initial curves, the integrals  $\int_{\mathbb{S}_j} k^2 ds$  and  $|\partial_x \sigma_j^i(x)|$  (from above and away from zero) for all the approximating networks are equibounded, thus Proposition 5.16 assures the existence of a uniform interval  $[0, T)$  of existence of smooth evolutions given by the curves  $\gamma_j^i(x, t) : [0, 1] \times [0, T) \rightarrow \overline{\Omega}$ .

Now, by the same reason, Proposition 5.12 gives uniform estimates on the  $L^\infty$  norms of the curvature and of all its derivatives in every rectangle  $[0, 1] \times [\mu, T_M)$ , with  $\mu > 0$ .

This means that if we reparametrize at every time all the curves  $\gamma_j^i$  proportional to their arclength, by means of a diagonal argument, we can find a subsequence of the family of reparametrized flows  $\tilde{\gamma}_j^i$  which converges in  $C_{\text{loc}}^\infty([0, 1] \times (0, T))$  to some flow, parametrized proportional to its arclength,  $\tilde{\gamma}^i$  in the time interval  $(0, T)$ . Moreover, by the hypotheses, the curves of the initial networks  $\tilde{\sigma}_j^i$  converge in  $W^{2,2}(0, 1)$  to  $\tilde{\sigma}^i$  which are the reparametrizations, proportional to their arclength, of the curves  $\sigma^i$  of the initial network  $\mathbb{S}_0$ . If we show that the maps  $\tilde{\gamma}^i$  are continuous up to the time  $t = 0$  we have a curvature flow for the network  $\tilde{\mathbb{S}}_0 = \bigcup_{i=1}^n \tilde{\sigma}^i([0, 1])$  which then gives a curvature flow for the original network  $\mathbb{S}_0$  in  $C^\infty([0, 1] \times (0, T))$ , reparametrizing it back with some family of continuous maps  $\theta^i : [0, 1] \times [0, T) \rightarrow [0, 1]$  with  $\theta_x^i \neq 0$  everywhere,  $\theta^i \in C^\infty([0, 1] \times (0, T))$  and  $\tilde{\sigma}^i(\theta^i(\cdot, 0)) = \sigma^i$  (this can be easily done as the maps  $\theta^i(\cdot, 0)$  are of class  $C^2$ , since in general, the arclength reparametrization maps have the same regularity of the network).

Hence, we deal with the continuity up to  $t = 0$  of the maps  $\tilde{\gamma}^i$ . By the uniform  $L^2$  bound on the curvature and the parametrization proportional to the arclength, the theorem of Ascoli–Arzelà implies that for every sequence of times  $t_l \rightarrow 0$ , the curves  $\tilde{\gamma}^i(\cdot, t_l)$  have a converging subsequence in  $C^1([0, 1])$  to some family of limit curves  $\zeta^i : [0, 1] \rightarrow \overline{\Omega}$ , still parametrized proportionally to arclength, by the  $C^1$ -convergence. Moreover, we can also assume that  $k(\cdot, t_l)$  converge weakly in  $L^2(ds)$  to the curvature function associated to the family of curves  $\zeta^i$ . We want to see that actually  $\zeta^i = \tilde{\sigma}^i$ , hence showing that the flow  $\tilde{\gamma}^i : [0, 1] \times [0, T) \rightarrow \overline{\Omega}$  is continuous and that the unit tangent vector  $\tau : [0, 1] \times [0, T) \rightarrow \mathbb{R}^2$  is a continuous map up to the time  $t = 0$  (this property is stable under the above reparametrization so it then will hold also for the final curvature flow  $\gamma^i$ ).

We consider a function  $\varphi \in C^\infty(\mathbb{R}^2)$  and the time derivative of its integral on the evolving networks  $\tilde{\gamma}_j^i$ , that is,

$$\begin{aligned} \frac{d}{dt} \int_{\tilde{\mathbb{S}}_j(t)} \varphi ds &= \int_{\tilde{\mathbb{S}}_j(t)} \varphi(\tilde{\lambda}_s - \tilde{k}^2) ds + \int_{\tilde{\mathbb{S}}_j(t)} \langle \nabla \varphi | \tilde{\underline{k}} + \tilde{\underline{\lambda}} \rangle ds \\ &= - \int_{\tilde{\mathbb{S}}_j(t)} \varphi \tilde{k}^2 ds - \int_{\tilde{\mathbb{S}}_j(t)} \langle \nabla \varphi | \tilde{\tau} \rangle \tilde{\lambda} ds + \int_{\tilde{\mathbb{S}}_j(t)} \langle \nabla \varphi | \tilde{\underline{k}} + \tilde{\underline{\lambda}} \rangle ds \\ &= - \int_{\tilde{\mathbb{S}}_j(t)} \varphi \tilde{k}^2 ds + \int_{\tilde{\mathbb{S}}_j(t)} \langle \nabla \varphi | \tilde{\underline{k}} \rangle ds, \end{aligned}$$

where we integrated by parts, passing from first to second line.

Let us consider now any sequence of times  $t_l$  converging to zero as above, such that the curves  $\tilde{\gamma}^i(\cdot, t_l)$  converge in  $C^1([0, 1])$  to some family of limit curves  $\zeta^i : [0, 1] \rightarrow \bar{\Omega}$  (still parametrized proportionally to arclength) as above, describing some regular network  $\bar{\mathbb{S}}$ , and  $k(\cdot, t_l)$  converge weakly in  $L^2(ds)$  to the curvature function associated to the family of curves  $\zeta^i$ . Integrating this equality in the time interval  $[0, t_l]$  we get

$$\int_{\tilde{\mathbb{S}}_j(t_l)} \varphi ds - \int_{\tilde{\mathbb{S}}_j(0)} \varphi ds = - \int_0^{t_l} \int_{\tilde{\mathbb{S}}_j(t)} \varphi \tilde{k}^2 ds dt + \int_0^{t_l} \int_{\tilde{\mathbb{S}}_j(t)} \langle \nabla \varphi | \tilde{\underline{k}} \rangle ds dt$$

which clearly passes to the limit as  $j \rightarrow \infty$ , by the smooth convergence of the flows  $\tilde{\gamma}_j^i$  to the flow  $\tilde{\gamma}^i$  (and the uniform bound on  $\int_{\tilde{\mathbb{S}}_j(t)} \tilde{k}^2 ds$ ) and of the initial networks  $\tilde{\mathbb{S}}_j(0) = \bigcup_{i=1}^n \tilde{\sigma}_j^i([0, 1])$  to  $\bar{\mathbb{S}}_0 = \bigcup_{i=1}^n \tilde{\sigma}^i([0, 1])$ , hence,

$$\int_{\tilde{\mathbb{S}}_{t_l}} \varphi ds - \int_{\bar{\mathbb{S}}_0} \varphi ds = - \int_0^{t_l} \int_{\tilde{\mathbb{S}}_t} \varphi \tilde{k}^2 ds dt + \int_0^{t_l} \int_{\tilde{\mathbb{S}}_t} \langle \nabla \varphi | \tilde{\underline{k}} \rangle ds dt$$

By the uniform bound on the  $L^2$  norm of the curvature, we then get

$$\left| \int_{\tilde{\mathbb{S}}_{t_l}} \varphi(\tilde{\gamma}(\cdot, t_l)) ds - \int_{\bar{\mathbb{S}}_0} \varphi(\tilde{\sigma}) ds \right| \leq C t_l,$$

where we made explicit the integrands, for sake of clarity. Sending  $l \rightarrow \infty$  we finally obtain

$$\left| \int_{\bar{\mathbb{S}}} \varphi(\zeta) ds - \int_{\bar{\mathbb{S}}_0} \varphi(\tilde{\sigma}) ds \right| = 0,$$

that is,

$$\int_{\bar{\mathbb{S}}} \varphi ds = \int_{\bar{\mathbb{S}}_0} \varphi ds$$

for every function  $\varphi \in C^\infty(\mathbb{R}^2)$ .

Since, both the networks  $\bar{\mathbb{S}}_0 = \bigcup_{i=1}^n \tilde{\sigma}^i([0, 1])$  and  $\bar{\mathbb{S}} = \bigcup_{i=1}^n \zeta^i([0, 1])$  are  $C^1$ , regular and parametrized proportionally to their arclength, this equality for every  $\varphi \in C^\infty(\mathbb{R}^2)$  implies that  $\tilde{\sigma}^i = \zeta^i$ , which is what we wanted.

Notice that, the continuity of  $\gamma^i$  and  $\tau$  also implies that the measures  $\mathcal{H}^1 \llcorner \mathbb{S}_t$  weakly\* converge to  $\mathcal{H}^1 \llcorner \bar{\mathbb{S}}_0$ , where  $\mathcal{H}^1$  is the one-dimensional Hausdorff measure, as  $t \rightarrow 0$ .

Finally, integrating on  $[0, t]$  inequality (5.8) (forgetting the absolute value and the contributions from the end-points), for the approximating flows  $\tilde{\gamma}_j^i$ , and passing to the limit as  $j \rightarrow \infty$ , we see that the function  $\int_{\tilde{\mathbb{S}}_t} k^2 ds$  is *continuous* on  $[0, T)$  (also at  $t = 0$ ), by the uniform bound on the  $L^2$  norm of the curvature of the networks. Being such integral invariant by reparametrization, this also holds for the flow  $\gamma^i$ . The same for the weak convergence in  $L^2(ds)$  of the functions  $k(\cdot, t)$  to  $k(\cdot, 0)$  as  $t \rightarrow 0$ .  $\square$

*Remark 6.9.*

1. The relevance of this theorem is that the initial network is not required to satisfy any compatibility condition, but only to have angles of 120 degrees between the concurring curves at every 3-point, that is, to be regular. In particular, it is not necessary that the sum of the three curvatures at a 3-point is zero.

2. The geometric uniqueness of the solution  $\gamma^i$  found in this theorem is an open problem.
3. As for every positive time the flow obtained by this theorem is  $C^\infty$ , hence every network  $\mathbb{S}_t$  is geometrically smooth, arguing as before (by means of Remark 4.24), the same conclusions of Corollary 6.2 apply, that is, this flow can be reparametrized, from any positive time on, to be a  $C^\infty$  special smooth flow.
4. It should be noticed that if the initial curves  $\sigma^i$  are  $C^\infty$ , the flow  $\mathbb{S}_t$  is smooth till  $t = 0$  far from the 3-points, that is, in any closed rectangle included in  $(0, 1) \times [0, T)$  we can locally reparametrize the curves  $\gamma^i$  to get a smooth flow up to  $t = 0$ . This follows from the local estimates for the motion by curvature (see [26]).
5. It is easy to see that, pushing a little the argument in the proof of this theorem, one can find a curvature flow with the same properties also if the initial network  $\mathbb{S}_0$  is regular and composed of regular curves of class  $W^{2,2}(0, 1)$  only.

Arguing as for Theorem 6.7, we have the following corollary.

**Corollary 6.10.** *If  $T < +\infty$  is the maximal time interval of existence of the curvature flow  $\mathbb{S}_t$  of an initial regular  $C^2$  network given by the previous theorem, then*

1. *either the inferior limit of the length of at least one curve of  $\mathbb{S}_t$  is zero, as  $t \rightarrow T$ ,*
2. *or  $\overline{\lim}_{t \rightarrow T} \int_{\mathbb{S}_t} k^2 ds = +\infty$ , hence, the curvature is not bounded as  $t \rightarrow T$ .*

*Moreover, if the lengths of the  $n$  curves are uniformly positively bounded from below, then this superior limit is actually a limit and there exists a positive constant  $C$  such that*

$$\int_{\mathbb{S}_t} k^2 ds \geq \frac{C}{\sqrt{T-t}} \quad \text{and} \quad \max_{\mathbb{S}_t} k^2 \geq \frac{C}{\sqrt{T-t}}$$

*for every  $t \in [0, T)$ .*

**Open Problem 6.11.** Every curvature flow of a regular network, hence only  $C^{2,1}$ , shares the properties stated in this corollary?

Notice that it would follow by a positive answer to Problem 6.5.

Now that we have gained a short time existence result for an initial regular  $C^2$  network, the next important question is what can be said if the initial network does not satisfy the 120 degrees condition, that is, it is non-regular (even if all its curves are  $C^\infty$ ). We will face this question in Section 11 below. Clearly, the unit tangent vectors of any curvature flow having as an initial network a configuration that does not satisfy the 120 degrees condition cannot be continuous up to time  $t = 0$ , being a curvature flow  $C^2$  and regular for positive time. Anyway, notice that in the definition of curvature flow we require only that the maps  $\gamma^i$  are continuous in  $[0, 1] \times (0, T)$  for some positive time  $T$ , hence one could hope to be able to find a curvature flow such that the 120 degrees condition is satisfied instantaneously, at every positive time  $t > 0$ , as it happens for the geometrical smoothness in Theorem 6.1.

In Section 11 we treat also the problem of the evolution of a non-regular network with multi-points of order greater than three. In this case even the meaning of the continuity condition of the maps at  $t = 0$  has to be redefined, since if we want that for every positive time the curvature flow is regular, actually the set of maps describing the network must change like its whole structure.

To deal with this situation, which is necessary also in order to be able to continue the flow when at some time a curve collapses and possibly some multi-points appear in the (limit) network, we need a more general (*weak*) and suitable definition of curvature flow.

As mentioned in the introduction, there exist several weak definitions of motion by curvature of a subset of  $\mathbb{R}^n$ . Among the existing notions, the most suitable to our point of view is the one of Brakke introduced in [13], which in general lacks uniqueness but at least maintains the (Hausdorff) dimension of the evolving sets.

## 6.1 Smooth flows are Brakke flows

We introduce now the concept of *Brakke flow* (with equality) of a network.

**Definition 6.12.** A *regular Brakke flow* is a family of  $W_{\text{loc}}^{2,2}$  networks  $\mathbb{S}_t$  in  $\Omega$ , satisfying the inequality

$$\frac{\overline{d}}{dt} \int_{\mathbb{S}_t} \varphi(\gamma, t) ds \leq - \int_{\mathbb{S}_t} \varphi(\gamma, t) k^2 ds + \int_{\mathbb{S}_t} \langle \nabla \varphi(\gamma, t) | \underline{k} \rangle ds + \int_{\mathbb{S}_t} \varphi_t(\gamma, t) ds, \quad (6.1)$$

for every non negative smooth function with compact support  $\varphi : \Omega \times [0, T) \rightarrow \mathbb{R}$  and  $t \in [0, T)$ , where  $\frac{\overline{d}}{dt}$  is the upper derivative (the  $\overline{\lim}$  of the incremental ratios).

If the time derivative at the left hand side exists and the inequality is an equality, for every smooth function with compact support  $\varphi : \Omega \times [0, T) \rightarrow \mathbb{R}$  and  $t \in [0, T)$ , that is,

$$\frac{d}{dt} \int_{\mathbb{S}_t} \varphi(\gamma, t) ds = - \int_{\mathbb{S}_t} \varphi(\gamma, t) k^2 ds + \int_{\mathbb{S}_t} \langle \nabla \varphi(\gamma, t) | \underline{k} \rangle ds + \int_{\mathbb{S}_t} \varphi_t(\gamma, t) ds, \quad (6.2)$$

we say that  $\mathbb{S}_t$  is a regular Brakke flow *with equality*.

*Remark 6.13.* Actually, the original definition of Brakke flow given in [13, Section 3.3] (in any dimension and codimension) allows the networks  $\mathbb{S}_t$  to be simply one-dimensional countably rectifiable subsets of  $\mathbb{R}^2$ , with possible integer multiplicity  $\theta_t : \mathbb{S}_t \rightarrow \mathbb{N}$ , and with a distributional notion of tangent space and (mean) curvature, called *rectifiable varifolds* (see [77]). With such a general definition, the networks are identified with the associated Radon measures  $\mu_t = \theta_t \mathcal{H}^1 \llcorner \mathbb{S}_t$ .

More precisely, the inequality

$$\begin{aligned} \frac{\overline{d}}{dt} \int_{\mathbb{S}_t} \varphi(x, t) \theta_t(x) d\mathcal{H}^1(x) &\leq - \int_{\mathbb{S}_t} \varphi(x, t) k^2(x, t) \theta_t(x) d\mathcal{H}^1(x) + \int_{\mathbb{S}_t} \langle \nabla \varphi(x, t) | \underline{k}(x, t) \rangle \theta_t(x) d\mathcal{H}^1(x) \\ &\quad + \int_{\mathbb{S}_t} \varphi_t(x, t) \theta_t(x) d\mathcal{H}^1(x), \end{aligned}$$

must hold for every non negative smooth function with compact support  $\varphi : \Omega \times [0, T) \rightarrow \mathbb{R}$  and  $t \in [0, T)$ , where  $\mathcal{H}^1$  is the Hausdorff one-dimensional measure in  $\mathbb{R}^2$ .

These weak conditions were introduced by Brakke in order to prove an existence result [13, Section 4.13] for a family of initial sets much wider than networks of curves, but, on the other hand, it lets open the possibility of instantaneous vanishing of some parts of the sets during the evolution.

A big difference between Brakke flows and the evolutions obtained as solutions of Problem (2.3) is that the former networks are simply considered as *subsets* of  $\mathbb{R}^2$  without any mention to their parametrization (that clearly is not unique). This means that actually a Brakke flow can be a family of networks given by the maps  $\gamma^i(x, t)$  which are  $C^2$  in space, but possibly do not have absolutely any regularity with respect to the time variable  $t$ .

An open question is whether any Brakke flow with equality, possibly under some extra hypotheses, admits a reparametrization such that it becomes a solution of Problem (2.3).

This problem is clearly also related to the uniqueness of the Brakke flows with equality (maybe further restricting the candidates to a special class with extra geometric properties).

**Proposition 6.14.** Any solution of Problem (2.3) in  $C^{2,1}([0, 1] \times [0, T))$  is a regular Brakke flow with equality. In particular, for every curve  $\gamma^i(\cdot, t)$  and for every time  $t \in [0, T)$  we have

$$\frac{dL^i(t)}{dt} = \lambda^i(1, t) - \lambda^i(0, t) - \int_{\gamma^i(\cdot, t)} k^2 ds \quad (6.3)$$

and

$$\frac{dL(t)}{dt} = - \int_{\mathbb{S}_t} k^2 ds,$$

that is, the total length  $L(t)$  is decreasing in time and it is uniformly bounded by the length of the initial network  $\mathbb{S}_0$ .

*Proof.* If the flow  $\gamma^i$  is in  $C^\infty([0, 1] \times [0, T])$ , we have

$$\begin{aligned}
\frac{dL^i(t)}{dt} &= \frac{d}{dt} \int_0^1 |\gamma_x^i| dx \\
&= \int_0^1 \frac{\langle \gamma_{xt}^i | \gamma_x^i \rangle}{|\gamma_x^i|} dx \\
&= \int_0^1 \left\langle \partial_x \gamma_t^i \left| \frac{\gamma_x^i}{|\gamma_x^i|} \right. \right\rangle dx \\
&= \int_0^1 \langle \partial_x \gamma_t^i | \tau^i \rangle dx \\
&= \langle \gamma_t^i(1, t) | \tau^i(1, t) \rangle - \langle \gamma_t^i(0, t) | \tau^i(0, t) \rangle - \int_0^1 \langle \gamma_t^i | \partial_x \tau^i \rangle dx.
\end{aligned}$$

Then, approximating the maps  $\gamma^i$  with a family of maps  $\gamma^{i\varepsilon} \in C^\infty$  such that  $\gamma^{i\varepsilon} \rightarrow \gamma^i$  in  $C^1$  and  $\gamma_{xx}^{i\varepsilon} \rightarrow \gamma_{xx}^i$  in  $C^0$ , as  $\varepsilon \rightarrow 0$ , we see that we can pass to the limit in this formula and conclude that it holds for the original flow which is only in  $C^{2,1}([0, 1] \times [0, T])$ . Finally, since  $\partial_x \tau^i = k^i \nu^i |\gamma_x^i|$ , we get

$$\frac{dL^i(t)}{dt} = \lambda^i(1, t) - \lambda^i(0, t) - \int_{\gamma^i(\cdot, t)} k^2 ds$$

as  $\gamma_t^i = k^i \nu^i + \lambda^i \tau^i$ .

The formula for the derivative of the total length of the evolving network then follows by the zero-sum property of the functions  $\lambda^i$  at every 3-point at the fact that all the  $\lambda^i$  are zero at the end-points.

A similar argument shows that formula (6.2) defining a regular Brakke flow with equality also holds.  $\square$

**Theorem 6.15.** *If  $\mathbb{S}_t$  is a curvature flow of a  $C^2$  initial network such that*

- *the unit tangents  $\tau^i$  are continuous in  $[0, 1] \times [0, T]$ ,*
- *the functions  $k(\cdot, t)$  converge weakly in  $L^2$  to  $k(\cdot, 0)$ , as  $t \rightarrow 0$ ,*
- *the function  $\int_{\mathbb{S}_t} k^2 ds$  is continuous on  $[0, T]$ ,*

*then  $\mathbb{S}_t$  is a regular Brakke flow with equality.*

*Proof.* By the previous Theorem 6.14, we only need to check Brakke equality (6.2) at  $t = 0$ .

For every positive time and for every smooth test function  $\varphi : \overline{\Omega} \times [0, T) \rightarrow \mathbb{R}$ , we have

$$\frac{d}{dt} \int_{\mathbb{S}_t} \varphi ds = - \int_{\mathbb{S}_t} \varphi k^2 ds + \int_{\mathbb{S}_t} \langle \nabla \varphi | \underline{k} \rangle ds + \int_{\mathbb{S}_t} \varphi_t ds,$$

hence, it suffices to show that the right member is continuous at  $t = 0$ . By the hypotheses, the only term that really need to be checked is  $\int_{\mathbb{S}_t} \varphi k^2 ds$ , we separate it as the sum of  $\int_{\mathbb{S}_t} \varphi^+ k^2 ds$  and  $\int_{\mathbb{S}_t} \varphi^- k^2 ds$  and we show the continuity of these two terms separately (here  $\varphi^+ = \varphi \wedge 0$  and  $\varphi^- = \varphi \vee 0$ ). Thus, we assume that  $0 \leq \varphi \leq 1$ , then, by the weak convergence in  $L^2(ds)$  of  $k(\cdot, t)$  to  $k(\cdot, 0)$ , the integral  $\int_{\mathbb{S}_t} \varphi k^2 ds$  is lower semicontinuous in  $t$ , that is,  $\int_{\mathbb{S}_0} \varphi k^2 ds \leq \liminf_{t_l \rightarrow 0} \int_{\mathbb{S}_{t_l}} \varphi k^2 ds$  for every  $t_l \rightarrow 0$ , but if this is not an equality for some sequence of times, it cannot happen that  $\int_{\mathbb{S}_t} k^2 ds$  is continuous at  $t = 0$ , indeed, we would have

$$\begin{aligned}
\lim_{t_l \rightarrow 0} \int_{\mathbb{S}_{t_l}} k^2 ds &\geq \liminf_{t_l \rightarrow 0} \int_{\mathbb{S}_{t_l}} \varphi k^2 ds + \liminf_{t_l \rightarrow 0} \int_{\mathbb{S}_{t_l}} (1 - \varphi) k^2 ds \\
&> \int_{\mathbb{S}_0} \varphi k^2 ds + \int_{\mathbb{S}_0} (1 - \varphi) k^2 ds = \int_{\mathbb{S}_0} k^2 ds.
\end{aligned}$$

This concludes the proof.  $\square$

**Corollary 6.16.** *The curvature flows whose short time existence is proved in Theorems 4.13 and 4.20 are regular Brakke flows with equality. The curvature flow of an initial  $C^2$  regular network obtained in Theorem 6.8 is also a regular Brakke flow with equality. Any curvature flow of a regular network is a regular Brakke flow with equality for every positive time.*

We conclude this section with the following property of Brakke flows.

**Proposition 6.17.** *For any regular Brakke flow with equality (hence, for every curvature flow of a regular network) such that the curvature is uniformly bounded in a time interval  $[0, T)$ , the lengths of the curves of the network  $L^i(t)$  converge to some limit, as  $t \rightarrow T$ .*

*In particular, if the flow satisfies the conclusion of Theorem 6.7 or Corollary 6.10 at the maximal time of existence  $T$ , there must be at least one curve such that  $L^i(t) \rightarrow 0$ , as  $t \rightarrow T$ .*

*Proof.* If the curvature is bounded, by formula (6.3), any function  $L^i$  as a uniformly bounded derivative, as  $k$  controls  $\lambda$  at the end-points and 3-points of the network, thus the conclusion follows.  $\square$

## 7 The monotonicity formula and rescaling procedures

Let  $F : \mathbb{S} \times [0, T) \rightarrow \mathbb{R}^2$  be the curvature flow of a regular network in its maximal time interval of existence. As before, with a little abuse of notation, we will write  $\tau(P^r, t)$  and  $\lambda(P^r, t)$  respectively for the unit tangent vector and the tangential velocity at the end-point  $P^r$  of the curve of the network getting at such point, for any  $r \in \{1, 2, \dots, l\}$ .

A modified form of Huisken's monotonicity formula for smooth hypersurfaces moving by mean curvature (see [42]), holds. It can be proved starting by formula (6.2) and with a slight modification of the computation in the proof of Lemma 6.3 in [63].

Let  $x_0 \in \mathbb{R}^2$ ,  $t_0 \in (0, +\infty)$  and  $\rho_{x_0, t_0} : \mathbb{R}^2 \times [0, t_0)$  be the one-dimensional backward heat kernel in  $\mathbb{R}^2$  relative to  $(x_0, t_0)$ , that is,

$$\rho_{x_0, t_0}(x, t) = \frac{e^{-\frac{|x-x_0|^2}{4(t_0-t)}}}{\sqrt{4\pi(t_0-t)}}.$$

We will often write  $\rho_{x_0}(x, t)$  to denote  $\rho_{x_0, T}(x, t)$  (or  $\rho_{x_0}$  to denote  $\rho_{x_0, T}$ ), when  $T$  is the maximal (singular) time of existence of a smooth curvature flow.

**Proposition 7.1** (Monotonicity formula). *Assume  $t_0 > 0$ . For every  $x_0 \in \mathbb{R}^2$  and  $t \in [0, \min\{t_0, T\})$  the following identity holds*

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{S}_t} \rho_{x_0, t_0}(x, t) ds &= - \int_{\mathbb{S}_t} \left| \underline{k} + \frac{(x-x_0)^\perp}{2(t_0-t)} \right|^2 \rho_{x_0, t_0}(x, t) ds \\ &\quad + \sum_{r=1}^l \left[ \left\langle \frac{P^r - x_0}{2(t_0-t)} \middle| \tau(P^r, t) \right\rangle - \lambda(P^r, t) \right] \rho_{x_0, t_0}(P^r, t). \end{aligned} \quad (7.1)$$

Integrating between  $t_1$  and  $t_2$  with  $0 \leq t_1 \leq t_2 < \min\{t_0, T\}$  we get

$$\begin{aligned} \int_{t_1}^{t_2} \int_{\mathbb{S}_t} \left| \underline{k} + \frac{(x-x_0)^\perp}{2(t_0-t)} \right|^2 \rho_{x_0, t_0}(x, t) ds dt &= \int_{\mathbb{S}_{t_1}} \rho_{x_0, t_0}(x, t_1) ds - \int_{\mathbb{S}_{t_2}} \rho_{x_0, t_0}(x, t_2) ds \\ &\quad + \sum_{r=1}^l \int_{t_1}^{t_2} \left[ \left\langle \frac{P^r - x_0}{2(t_0-t)} \middle| \tau(P^r, t) \right\rangle - \lambda(P^r, t) \right] \rho_{x_0, t_0}(P^r, t) dt. \end{aligned}$$

We need the following lemma in order to estimate the end-points contribution in this formula (see Lemma 6.5 in [63]).

**Lemma 7.2.** *For every  $r \in \{1, 2, \dots, l\}$  and  $x_0 \in \mathbb{R}^2$ , the following estimate holds*

$$\left| \int_t^{t_0} \left[ \left\langle \frac{P^r - x_0}{2(t_0-\xi)} \middle| \tau(P^r, \xi) \right\rangle - \lambda(P^r, \xi) \right] \rho_{x_0, t_0}(P^r, \xi) d\xi \right| \leq C,$$



where  $C$  is a constant depending only on the constants  $C_l$  in assumption (5.1).

As a consequence, for every point  $x_0 \in \mathbb{R}^2$ , we have

$$\lim_{t \rightarrow t_0} \sum_{r=1}^l \int_t^{t_0} \left[ \left\langle \frac{P^r - x_0}{2(t_0 - \xi)} \middle| \tau(P^r, \xi) \right\rangle - \lambda(P^r, \xi) \right] \rho_{x_0, t_0}(P^r, \xi) d\xi = 0.$$

As a consequence, the following definition is well posed.

**Definition 7.3** (Gaussian densities). For every  $x_0 \in \mathbb{R}^2, t_0 \in (0, +\infty)$  we define the *Gaussian density* function  $\Theta_{x_0, t_0} : [0, \min\{t_0, T\}) \rightarrow \mathbb{R}$  as

$$\Theta_{x_0, t_0}(t) = \int_{\mathbb{S}_t} \rho_{x_0, t_0}(\cdot, t) ds$$

and, provided  $t_0 \leq T$ , the *limit Gaussian density* function  $\hat{\Theta} : \mathbb{R}^2 \times (0, +\infty) \rightarrow \mathbb{R}$  as

$$\hat{\Theta}(x_0, t_0) = \lim_{t \rightarrow t_0} \Theta_{x_0, t_0}(t).$$

Moreover, we will often write  $\Theta_{x_0}(t)$  to denote  $\Theta_{x_0, T}(t)$  and  $\hat{\Theta}(x_0)$  for  $\hat{\Theta}(x_0, T)$ .

For every  $(x_0, t_0) \in \mathbb{R}^2 \times (0, T]$ , the limit  $\hat{\Theta}(x_0, t_0)$  exists, by the monotonicity of  $\Theta_{x_0, t_0}$ , and it is finite and non negative. Moreover, the map  $\hat{\Theta} : \mathbb{R}^2 \rightarrow \mathbb{R}$  is upper semicontinuous (see [59, Proposition 2.12]).

## 7.1 Parabolic rescaling of the flow

For a fixed  $\mu > 0$  the standard parabolic rescaling of a curvature flow given by the map  $F$  above, around a space-time point  $(x_0, t_0)$ , is defined as the family of maps

$$F_t^\mu = \mu(F(\cdot, \mu^{-2}t + t_0) - x_0), \quad (7.2)$$

where  $t \in [-\mu^2 t_0, \mu^2(T - t_0)]$ . Note that this is again a curvature flow in the domain  $\mu(\Omega - x_0)$  with new time parameter  $t$ .

Given a sequence  $\mu_i \nearrow +\infty$  and a space-time point  $(x_0, t_0)$ , where  $0 < t_0 \leq T$ , we then consider the sequence of curvature flows  $F_t^{\mu_i}$  in the whole  $\mathbb{R}^2$  that we denote with  $\mathbb{S}_t^{\mu_i}$ .

Recall that the monotonicity formula implies

$$\begin{aligned} \Theta_{x_0, t_0}(t) - \hat{\Theta}(x_0, t_0) &= \int_t^{t_0} \int_{\mathbb{S}_\sigma} \left| \underline{k} + \frac{(x - x_0)^\perp}{2(t_0 - \sigma)} \right|^2 \rho_{x_0, t_0}(\cdot, \sigma) ds d\sigma \\ &\quad - \sum_{r=1}^l \int_t^{t_0} \left[ \left\langle \frac{P^r - x_0}{2(t_0 - \sigma)} \middle| \tau(P^r, \sigma) \right\rangle - \lambda(P^r, \sigma) \right] \rho_{x_0, t_0}(P^r, \sigma) d\sigma. \end{aligned}$$

Changing variables according to the parabolic rescaling, we obtain

$$\begin{aligned} \Theta_{x_0, t_0}(t_0 + \mu_i^{-2}t) - \hat{\Theta}(x_0, t_0) &= \int_t^0 \int_{\mathbb{S}_s^{\mu_i}} \left| \underline{k} - \frac{x^\perp}{2s} \right|^2 \rho_{0,0}(\cdot, s) ds ds \\ &\quad + \sum_{r=1}^l \int_t^0 \left[ \left\langle \frac{P_i^r}{2s} \middle| \tau(P_i^r, s) \right\rangle + \lambda(P_i^r, s) \right] \rho_{0,0}(P_i^r, s) ds, \end{aligned}$$

where  $P_i^r = \mu_i(P^r - x_0)$ .

Hence, sending  $i \rightarrow \infty$ , by Lemma 7.2, for every  $t \in (-\infty, 0)$  we get

$$\lim_{i \rightarrow \infty} \int_t^0 \int_{\mathbb{S}_s^{\mu_i}} \left| \underline{k} - \frac{x^\perp}{2s} \right|^2 \rho_{0,0}(\cdot, s) ds ds = 0.$$

## 7.2 Huisken's dynamical rescaling

Next, we introduce the rescaling procedure of Huisken in [42] at the maximal time  $T$ .

Fixed  $x_0 \in \mathbb{R}^2$ , let  $\tilde{F}_{x_0} : \mathbb{S} \times [-1/2 \log T, +\infty) \rightarrow \mathbb{R}^2$  be the map

$$\tilde{F}_{x_0}(p, t) = \frac{F(p, t) - x_0}{\sqrt{2(T-t)}} \quad t(t) = -\frac{1}{2} \log(T-t)$$

then, the rescaled networks are given by

$$\tilde{\mathbb{S}}_{x_0, t} = \frac{\mathbb{S}_t - x_0}{\sqrt{2(T-t)}} \quad (7.3)$$

and they evolve according to the equation

$$\frac{\partial}{\partial t} \tilde{F}_{x_0}(p, t) = \tilde{\underline{v}}(p, t) + \tilde{F}_{x_0}(p, t)$$

where

$$\tilde{\underline{v}}(p, t) = \sqrt{2(T-t(t))} \cdot \underline{v}(p, t(t)) = \tilde{\underline{k}} + \tilde{\underline{\lambda}} = \tilde{k}\nu + \tilde{\lambda}\tau \quad \text{and} \quad t(t) = T - e^{-2t}.$$

Notice that we did not put the sign over the unit tangent and normal, since they remain the same after the rescaling.

We will write  $\tilde{O}^p(t) = \tilde{F}_{x_0}(O^p, t)$  for the 3-points of the rescaled network  $\tilde{\mathbb{S}}_{x_0, t}$  and  $\tilde{P}^r(t) = \tilde{F}_{x_0}(P^r, t)$  for the end-points, when there is no ambiguity on the point  $x_0$ .

The rescaled curvature evolves according to the following equation,

$$\partial_t \tilde{k} = \tilde{k}_{\mathfrak{s}\mathfrak{s}} + \tilde{k}_{\mathfrak{s}} \tilde{\lambda} + \tilde{k}^3 - \tilde{k}$$

which can be obtained by means of the commutation law

$$\partial_t \partial_{\mathfrak{s}} = \partial_{\mathfrak{s}} \partial_t + (\tilde{k}^2 - \tilde{\lambda}_{\mathfrak{s}} - 1) \partial_{\mathfrak{s}},$$

where we denoted with  $\mathfrak{s}$  the arclength parameter for  $\tilde{\mathbb{S}}_{x_0, t}$ .

*Remark 7.4.* It is easy to see that the relations between the two rescaling procedures are given by

$$\mathbb{S}_t^\mu = \sqrt{-2t} \tilde{\mathbb{S}}_{x_0, \log(\mu/\sqrt{-t})} \quad \text{and} \quad \tilde{\mathbb{S}}_{x_0, t} = \frac{e^t}{\mu\sqrt{2}} \mathbb{S}_{-\mu^2 e^{-2t}}^\mu,$$

in particular,

$$\mathbb{S}_{-1/2}^\mu = \tilde{\mathbb{S}}_{x_0, \log(\mu\sqrt{2})}.$$

By a straightforward computation (see [42]) we have the following rescaled version of the monotonicity formula.

**Proposition 7.5** (Rescaled monotonicity formula). *Let  $x_0 \in \mathbb{R}^2$  and set*

$$\tilde{\rho}(x) = e^{-\frac{|x|^2}{2}}$$

*For every  $t \in [-1/2 \log T, +\infty)$  the following identity holds*

$$\frac{d}{dt} \int_{\tilde{\mathbb{S}}_{x_0, t}} \tilde{\rho}(x) d\mathfrak{s} = - \int_{\tilde{\mathbb{S}}_{x_0, t}} |\tilde{\underline{k}} + x^\perp|^2 \tilde{\rho}(x) d\mathfrak{s} + \sum_{r=1}^l \left[ \left\langle \tilde{P}^r(t) \mid \tau(P^r, t(t)) \right\rangle - \tilde{\lambda}(P^r, t) \right] \tilde{\rho}(\tilde{P}^r(t))$$

*where  $\tilde{P}^r(t) = \frac{P^r - x_0}{\sqrt{2(T-t(t))}}$ .*

*Integrating between  $t_1$  and  $t_2$  with  $-1/2 \log T \leq t_1 \leq t_2 < +\infty$  we get*

$$\begin{aligned} \int_{t_1}^{t_2} \int_{\tilde{\mathbb{S}}_{x_0, t}} |\tilde{\underline{k}} + x^\perp|^2 \tilde{\rho}(x) d\mathfrak{s} dt &= \int_{\tilde{\mathbb{S}}_{x_0, t_1}} \tilde{\rho}(x) d\mathfrak{s} - \int_{\tilde{\mathbb{S}}_{x_0, t_2}} \tilde{\rho}(x) d\mathfrak{s} \\ &+ \sum_{r=1}^l \int_{t_1}^{t_2} \left[ \left\langle \tilde{P}^r(t) \mid \tau(P^r, t(t)) \right\rangle - \tilde{\lambda}(P^r, t) \right] \tilde{\rho}(\tilde{P}^r(t)) dt. \end{aligned} \quad (7.4)$$

We have also the analog of Lemma 7.2 (see Lemma 6.7 in [63]).

**Lemma 7.6.** *For every  $r \in \{1, 2, \dots, l\}$  and  $x_0 \in \mathbb{R}^2$ , the following estimate holds for all  $t \in [-\frac{1}{2} \log T, +\infty)$ ,*

$$\left| \int_t^{+\infty} \left[ \left\langle \tilde{P}^r(\xi) \mid \tau(P^r, t(\xi)) \right\rangle - \tilde{\lambda}(P^r, \xi) \right] d\xi \right| \leq C,$$

where  $C$  is a constant depending only on the constants  $C_l$  in assumption (5.1).

As a consequence, for every point  $x_0 \in \mathbb{R}^2$ , we have

$$\lim_{t \rightarrow +\infty} \sum_{r=1}^l \int_t^{+\infty} \left[ \left\langle \tilde{P}^r(\xi) \mid \tau(P^r, t(\xi)) \right\rangle - \tilde{\lambda}(P^r, \xi) \right] d\xi = 0.$$

## 8 Classification of possible blow-up limits

In this section we want to discuss the possible limits of an evolving network at the maximal time of existence. When the curvature does not remain bounded, we are interested in the possible blow-up limit networks after parabolic or Huisken's rescaling procedure, using the rescaled monotonicity formula (see Section 7). In some cases, such limit sets are no more regular networks, so we introduce the following definition.

**Definition 8.1** (Degenerate regular network). Consider a tuple  $(G, \mathbb{S})$  with the following properties:

- $G = \bigcup_{i=1}^n E^i$  is an oriented graph with possible unbounded edges  $E^i$ , such that every vertex has only one or three concurring edges (we call end-points of  $G$  the vertices with order one);
- given a family of  $C^1$  curves  $\sigma^i : I^i \rightarrow \mathbb{R}^2$ , where  $I^i$  is the interval  $(0, 1)$ ,  $[0, 1)$ ,  $(0, 1]$  or  $[0, 1]$ , and orientation preserving homeomorphisms  $\varphi^i : E^i \rightarrow I^i$ , then  $\mathbb{S}$  is the union of the images of  $I^i$  through the curves  $\sigma^i$ , that is,  $\mathbb{S} = \bigcup_{i=1}^n \sigma^i(I^i)$  (notice that the interval  $(0, 1)$  can only appear if it is associated to an unbounded edge  $E^i$  without vertices, which is clearly a single connected component of  $G$ );
- in the case that  $I^i$  is  $(0, 1)$ ,  $[0, 1)$  or  $(0, 1]$ , the map  $\sigma^i$  is a regular  $C^1$  curve with unit tangent vector field  $\tau^i$ ;
- in the case that  $I^i = [0, 1]$ , the map  $\sigma^i$  is either a regular  $C^1$  curve with unit tangent vector field  $\tau^i$ , or a constant map and in this case it is "assigned" also a *constant* unit vector  $\tau^i : I^i \rightarrow \mathbb{R}^2$ , that we still call unit tangent vector of  $\sigma^i$  (we call these maps  $\sigma^i$  "degenerate curves");
- for every degenerate curve  $\sigma^i : I^i \rightarrow \mathbb{R}^2$  with assigned unit vector  $\tau^i : I^i \rightarrow \mathbb{R}^2$ , we call "assigned exterior unit tangents" of the curve  $\sigma^i$  at the points 0 and 1 of  $I^i$ , respectively the unit vectors  $-\tau^i$  and  $\tau^i$ .
- the map  $\Gamma : G \rightarrow \mathbb{R}^2$  given by the union  $\Gamma = \bigcup_{i=1}^n (\sigma^i \circ \varphi^i)$  is well defined and continuous;
- for every 3-point of the graph  $G$ , where the edges  $E^i, E^j, E^k$  concur, the exterior unit tangent vectors (real or "assigned") at the relative borders of the intervals  $I^i, I^j, I^k$  of the concurring curves  $\sigma^i, \sigma^j, \sigma^k$  have zero sum ("degenerate 120 degrees condition").

Then, we call  $\mathbb{S} = \bigcup_{i=1}^n \sigma^i(I^i)$  a *degenerate regular network*.

If one or several edges  $E^i$  of  $G$  are mapped under the map  $\Gamma : G \rightarrow \mathbb{R}^2$  to a single point  $p \in \mathbb{R}^2$ , we call this sub-network given by the union  $G'$  of such edges  $E^i$ , the *core* of  $\mathbb{S}$  at  $p$ .

We call multi-points of the degenerate regular network  $\mathbb{S}$ , the images of the vertices of multiplicity three of the graph  $G$ , by the map  $\Gamma$ .

We call end-points of the degenerate regular network  $\mathbb{S}$ , the images of the vertices of multiplicity one of the graph  $G$ , by the map  $\Gamma$ .

*Remark 8.2.*

- A regular network is clearly a degenerate regular network.
- This definition will be useful to deal with the limit sets when at some time a curve of the network “collapses”, that is, its length goes to zero (later on in Section 10).
- A degenerate regular network  $\mathbb{S}$  with underlying graph  $G$ , seen as a subset in  $\mathbb{R}^2$ , is a  $C^1$  network, not necessarily regular, that can have end-points and/or unbounded curves. Moreover, self-intersections and curves with integer multiplicities can be present. Anyway, by the degenerate 120 degrees condition at the last point of the definition, at every image of a multi-point of  $G$  the sum (possibly with multiplicities) of the exterior unit tangents (the “assigned” ones cancel each other in pairs) is zero. Notice that this implies that every multiplicity-one 3-point must satisfy the 120 degrees condition.

**Lemma 8.3.** *Let  $\mathbb{S} = \bigcup_{i=1}^n \sigma^i(I_i)$  a degenerate regular network in  $\Omega$  and  $X : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a smooth vector field with compact support. Then, there holds*

$$\int_{\mathbb{S}} \partial_s \langle X(\sigma) | \tau \rangle d\overline{\mathcal{H}}^1 = - \sum_{r=1}^l \langle X(P^r) | \tau(P^r) \rangle,$$

where  $P^1, P^2, \dots, P^l$  are the end-points of  $\mathbb{S}$ ,  $\tau(P^1), \tau(P^2), \dots, \tau(P^l)$  are the exterior unit tangents at  $P^r$  and  $\overline{\mathcal{H}}^1$  is the one-dimensional Hausdorff measure, counting multiplicities.

*Proof.* This is a consequence of the degenerate 120 degrees condition, implying that the sum of all the contribution at a multi-point given by the boundary terms after the integration on each single curve, is zero (as the sum of the exterior unit tangents of the concurring curves). Thus, the only remaining terms are due to the end-points of the degenerate regular network.  $\square$

**Definition 8.4.** We say that a sequence of regular networks  $\mathbb{S}_k = \bigcup_{i=1}^n \sigma_k^i(I_k^i)$  converges in  $C_{\text{loc}}^1$  to a degenerate regular network  $\mathbb{S} = \bigcup_{j=1}^l \sigma_\infty^j(I_\infty^j)$  with underlying graph  $G = \bigcup_{j=1}^l E^j$  if:

- letting  $O^1, O^2, \dots, O^m$  the multi-points of  $\mathbb{S}$ , for every open set  $\Omega \subset \mathbb{R}^2$  with compact closure in  $\mathbb{R}^2 \setminus \{O^1, O^2, \dots, O^m\}$ , the networks  $\mathbb{S}_k$  restricted to  $\Omega$ , for  $k$  large enough, are described by families of regular curves which, after possibly reparametrizing them, converge to the family of regular curves given by the restriction of  $\mathbb{S}$  to  $\Omega$ ;
- for every multi-point  $O^p$  of  $\mathbb{S}$ , image of one or more vertices of the graph  $G$  (if a core is present), there is a sufficiently small  $R > 0$  and a graph  $\tilde{G} = \bigcup_{r=1}^s F^r$ , with edges  $F^r$  associated to intervals  $J^r$ , such that:
  - the restriction of  $\mathbb{S}$  to  $B_R(O^p)$  is a regular degenerate network described by a family of curves  $\tilde{\sigma}_\infty^r : J^r \rightarrow \mathbb{R}^2$  with (possibly “assigned”, if the curve is degenerate) unit tangent  $\tilde{\tau}_\infty^r$ ,
  - for  $k$  sufficiently large, the restriction of  $\mathbb{S}_k$  to  $B_R(O^p)$  is a regular network with underlying graph  $\tilde{G}$ , described by the family of regular curves  $\tilde{\sigma}_k^r : J^r \rightarrow \mathbb{R}^2$ ,
  - for every  $j$ , possibly after reparametrization of the curves, the sequence of maps  $J^r \ni x \mapsto (\tilde{\sigma}_k^r(x), \tilde{\tau}_k^r(x))$  converge in  $C_{\text{loc}}^0$  to the maps  $J^r \ni x \mapsto (\tilde{\sigma}_\infty^r(x), \tilde{\tau}_\infty^r(x))$ , for every  $r \in \{1, 2, \dots, s\}$ .

We will say that  $\mathbb{S}_k$  converges to  $\mathbb{S}$  in  $C_{\text{loc}}^1 \cap E$ , where  $E$  is some function space, if the above curves also converge in the topology of  $E$ .

*Remark 8.5.*

- It is easy to see that if a sequence of regular networks  $\mathbb{S}_k$  converges in  $C_{\text{loc}}^1$  to a degenerate regular network  $\mathbb{S}$ , the associated one-dimensional Hausdorff measures, counting multiplicities, weakly-converge (as measures) to the one-dimensional Hausdorff measure associated to the set  $\mathbb{S}$  seen as a subset of  $\mathbb{R}^2$ .
- If a degenerate regular network  $\mathbb{S}$  is the limit of a sequence of regular networks as above, being these embedded, it clearly can have only *tangent* self-intersections but not a “crossing” of two of its curves.

- If  $\mathbb{S}$  is the limit of a sequence of “rescalings” of the networks of a curvature flow  $\mathbb{S}_t$  with fixed end-points, it can have only one end-point at the origin of  $\mathbb{R}^2$  and only if the center of the rescalings coincides with an end-point of  $\mathbb{S}_t$ , otherwise, it has no end-points at all (they go to  $\infty$  in the rescaling).

## 8.1 Self-similarly shrinking networks

**Definition 8.6.** A regular  $C^2$  open network  $\mathbb{S} = \bigcup_{i=1}^n \sigma^i(I_i)$  is called a *regular shrinker* if at every point  $x \in \mathbb{S}$  there holds

$$\underline{k} + x^\perp = 0. \quad (8.1)$$

This relation is called the *shrinkers equation*.

The name comes from the fact that if  $\mathbb{S} = \bigcup_{i=1}^n \sigma^i(I_i)$  is a shrinker, then the evolution given by  $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i(I_i, t)$  where  $\gamma^i(x, t) = \sqrt{-2t} \sigma^i(x)$  is a self-similarly shrinking curvature flow in the time interval  $(-\infty, 0)$  with  $\mathbb{S} = \mathbb{S}_{-1/2}$ . Viceversa, if  $\mathbb{S}_t$  is a self-similarly shrinking curvature flow in the maximal time interval  $(-\infty, 0)$ , then  $\mathbb{S}_{-1/2}$  is a shrinker.

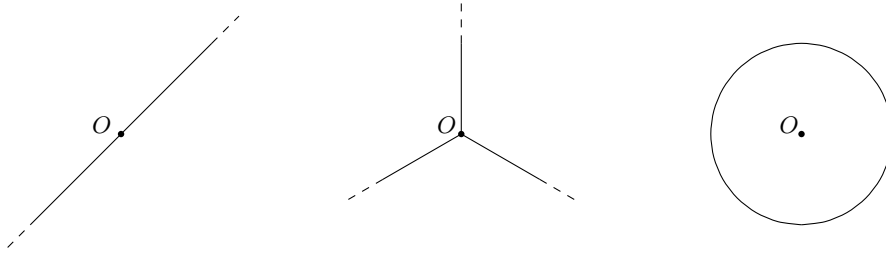


Figure 5: Easy examples of regular shrinkers are a line for the origin, an unbounded triod composed of three halflines from the origin meeting at 120 degrees, that we call *standard triod* and the unit circle  $\mathbb{S}^1$ .

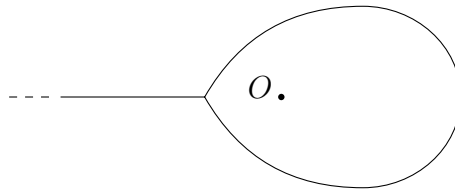


Figure 6: A less easy example of a regular shrinker: a *Brakke spoon*.

The regular shrinkers in the previous figures are all the ones with at most one triple junction (see [40]), in particular, by the work of Abresch and Langer [1], it follows that the only regular shrinkers *without* triple junctions (simply curves) are the lines for the origin and the unit circle. For shrinkers with two triple junctions, it is not difficult to show that there are only two possible topological shapes for a complete embedded, regular shrinker: one is the “lens/fish” shape and the other is the shape of the Greek “Theta” letter (or “double cell”), as in the next figure (see [9]).

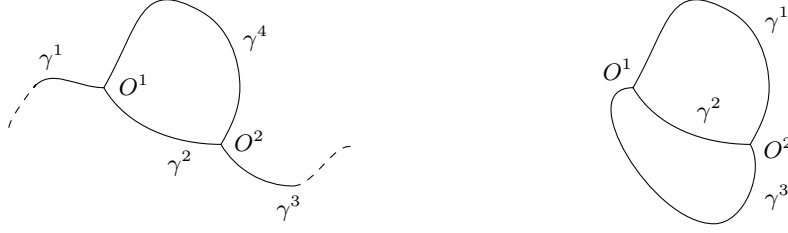


Figure 7: A lens/fish-shaped and a  $\Theta$ -shaped network.

It is well known that there exist unique (up to a rotation) lens-shaped or fish-shaped, embedded, regular shrinkers which are symmetric with respect to a line through the origin of  $\mathbb{R}^2$  (see [17, 74]). Instead, there are no regular  $\Theta$ -shaped shrinkers (see [10]).



Figure 8: A lens-shaped and a fish-shaped shrinker.

A “gallery” with these and other more complicated regular shrinkers can be found in the Appendix.

**Definition 8.7** (Degenerate shrinkers). We call a degenerate regular network  $\mathbb{S} = \bigcup_{i=1}^n \sigma^i(I_i)$  a *degenerate regular shrinker* if at every point  $x \in \mathbb{S}$  there holds

$$\underline{k} + x^\perp = 0.$$

Clearly, a regular shrinker is a degenerate regular shrinker and, as before, the maps  $\gamma^i(x, t) = \sqrt{-2t} \sigma^i(x)$  describe the self-similarly shrinking evolution of a degenerate regular network  $\mathbb{S}_t$  in the time interval  $(-\infty, 0)$ , with  $\mathbb{S} = \mathbb{S}_0$ .

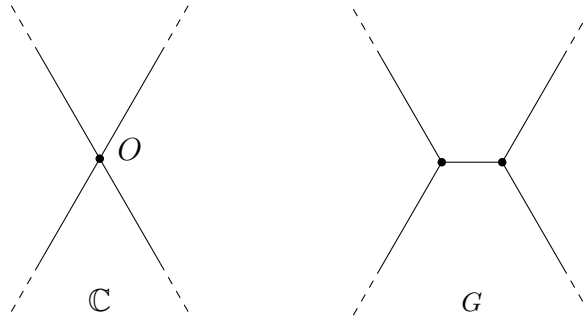


Figure 9: A so called *standard cross* with angles of 60/120 degrees and its underlying graph  $G$ .

**Remark 8.8.** As every non-degenerate curve of a degenerate regular shrinker (or simply of a regular shrinker) satisfies the equation  $\underline{k} + x^\perp = 0$ , it must be a piece of a line though the origin or of the so called *Abresch–Langer curves*. Their classification results in [1] imply that any of these non straight pieces is compact, hence any unbounded curve of a shrinker must be a line or an halfline “pointing” towards



the origin. Moreover, it also follows that if a curve contains the origin, then it is a straight line through the origin (if it is in the interior) or a halfline from the origin (if it is an end-point of the curve).

For a degenerate regular shrinker  $\mathbb{S}$ , in analogy with Definition 7.3, we denote with

$$\Theta_{\mathbb{S}} = \Theta_{0,0}(-1/2) = \int_{\mathbb{S}} \rho_{0,0}(\cdot, -1/2) d\bar{s}$$

its Gaussian density (here  $d\bar{s}$  denotes the integration with respect to the canonical measure on  $\mathbb{S}$ , counting multiplicities). Notice that the integral  $\Theta_{0,0}(t) = \int_{\mathbb{S}_t} \rho_{0,0}(\cdot, t) d\bar{s}$  is constant for  $t \in (-\infty, 0)$ , hence equal to  $\widehat{\Theta}(0)$  for the self-similarly shrinking curvature flow  $\mathbb{S}_t = \sqrt{-2t} \mathbb{S}$  generated by  $\mathbb{S}$ , as above.

The Gaussian density of a straight line through the origin is 1, of a halfline from the origin is  $1/2$ , of a standard triod  $\mathbb{T}$  is  $3/2$ , of a standard cross  $\mathbb{C}$  is 2. The Gaussian density of the unit circle  $\mathbb{S}^1$  can be easily computed to be

$$\Theta_{\mathbb{S}^1} = \sqrt{\frac{2\pi}{e}} \approx 1,5203.$$

Notice that  $\Theta_{\mathbb{T}} = 3/2 < \Theta_{\mathbb{S}^1} < 2$ .

The Gaussian densities of several other regular shrinkers can be found in the Appendix.

We have the following two classification results for degenerate regular shrinkers, see Lemma 8.3 and 8.4 in [47].

**Lemma 8.9.** *Let  $\mathbb{S} = \bigcup_{i=1}^n \sigma^i(I_i)$  be a degenerate regular shrinker which is  $C_{\text{loc}}^1$ -limit of regular networks homeomorphic to the underlying graph  $G$  of  $\mathbb{S}$  (as in Definition 8.1) and assume that  $G$  is a tree without end-points. Then  $\mathbb{S}$  consists of halflines from the origin, with possibly a core at the origin.*

*Moreover, if  $G$  is connected, without end-points and  $\mathbb{S}$  is a network with unit multiplicity, this latter can only be*

- *a line (no core),*
- *a standard triod (no core),*
- *two lines intersecting at the origin forming angles of 120/60 degrees (the core is a collapsed segment in the origin with “assigned” unit tangent vector bisecting the angles of 120 degrees), that is, a standard cross (see Figure 9).*

*Proof.* We assume that  $G$  is connected, otherwise we argue on every single connected component. By the hypothesis of approximation with regular (embedded) networks,  $G$  is a planar graph.

As we said in Remark 8.8, if a non-degenerate curve contains the origin, then it is a piece of a straight line. Otherwise, it is contained in a compact subset of  $\mathbb{R}^2$  and has a constant winding direction with respect to the origin. Aside from the circle, any other solution has a countable, non-vanishing number of self-intersections (all these facts were shown in [1]).

Suppose that the network  $\mathbb{S}$  has a core at some point  $P \in \mathbb{S}$ , then, at least an edge of  $G$  is mapped into  $P$ .

Being the graph  $G$  a tree, it can be seen easily by induction, that from  $P$  there must exit  $N + 2$  (not necessarily distinct) non-degenerate curves, where  $N$  is the number (greater than one) of 3-points contained in the core. Moreover, considering the longest simple “path” in  $G$  which is mapped in the core at  $P$  of  $\mathbb{S}$ , orienting it and “following” its edges, the assigned unit tangent vector (possibly changed of sign on some edges in order to coincide with the orientation of the path) cannot “turn” of an angle of 60 degrees in the same “direction” for two consecutive times, otherwise, since  $G$  is a tree, the approximating networks must have a self-intersection (see Figure 10 below).

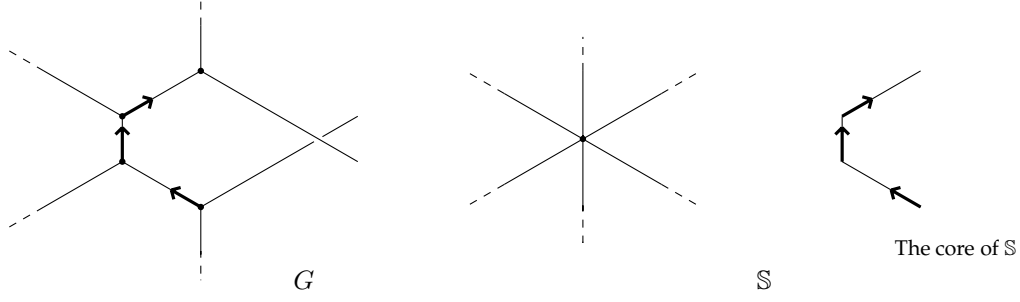


Figure 10: If the assigned unit tangent vector “turns” of an angle of 60 degrees in the same direction for two consecutive times,  $G$  has self-intersections. An example of such a pair  $(G, \mathbb{S})$ .

Hence, the assigned unit tangent vector “turns” of an angle of 60 degrees then it must “turn” back, in passing from an edge to another along such longest path. This means that at the initial/final point of such path, either the two assigned unit tangent vectors are the same (when the number of edges is odd) or they differ of 60 degrees (when the number of edges is even). By a simple check, we can then see that, in the first case the four curves images of the four non-collapsed edges exiting from such initial/final points of the path, have four different exterior unit tangent vectors at  $P$  (opposite in pairs), in the second case, they have three exterior unit tangent vectors at  $P$  which are non-proportional each other.

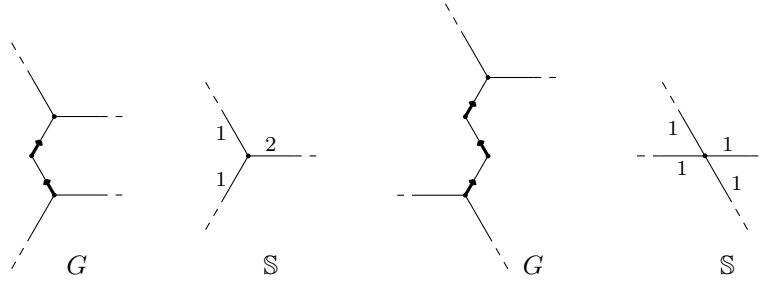


Figure 11: Examples of the edges at the initial/final points of the longest simple path in  $G$  and of the relative curves in  $\mathbb{S}$ , the numbers 1 and 2 denote their multiplicity.

If then there is a 3-point or a core at some point  $P \neq 0$ , since at most two of the four directions in the first case above and at most one of the three directions in the second case, can belong to the straight line for  $P$  and the origin, there are always at least two non-straight Abresch-Langer curves arriving/starting at  $P$ . Clearly, this property holds also if there is no core at  $P$ , but  $P$  is simply a 3-point.

Let us consider  $\mathbb{S}' \subset \mathbb{S}$ , which consists of  $\mathbb{S}$  with the interior of all the pieces of straight lines removed and let  $\sigma^i$  one of the two curves above. We follow  $\sigma^i$  till its other end-point  $Q$ . At this end-point, even if there is a core at  $Q$ , there is always another different non-straight curve  $\sigma^j$  to continue moving in  $\mathbb{S}$  avoiding the pieces of straight lines (hence staying far from the origin). Actually, either the underlying intervals  $I_i$  and  $I_j$  are concurrent at the vertex corresponding to  $Q$  in the graph  $G$  or there is a path in  $G$  (“collapsed” in the core at  $Q$ ) joining  $I_i$  and  $I_j$ . We then go on with this path on  $\mathbb{S}$  (and on  $G$ ) till, looking at things on the graph  $G$ , we arrive at an already considered vertex, which happens since the number of vertices of  $G$  is finite, obtaining a closed loop, hence, a contradiction. Thus,  $\mathbb{S}'$  cannot contain 3-points or cores outside the origin. If anyway  $\mathbb{S}$  contains a non-straight Abresch-Langer curve, we can repeat this argument getting again a contradiction, hence, we are done with the first part of the lemma, since then  $\mathbb{S}$  can only consist of halflines from the origin.

Now we assume that  $G$  is connected and  $\mathbb{S}$  is a network with multiplicity one, composed of halflines from the origin.

If there is no core,  $\mathbb{S}$  is homeomorphic to  $G$  and composed only by halflines for the origin, hence  $G$  has

at most one vertex, by connectedness. If  $G$  has no vertices, then  $\mathbb{S}$  must be a line, if it has a 3-point,  $\mathbb{S}$  is a standard triod.

If there is a core in the origin, by the definition of degenerate regular network it follows that the halflines of  $\mathbb{S}$  can only have six possible directions, by the 120 degrees condition, hence, by the unit multiplicity hypothesis, the graph  $G$  is a tree in the plane with at most six unbounded edges. Arguing as in the first part of the lemma, if  $N$  denotes the number (greater than one) of 3-points contained in the core, it follows that  $N$  can only assume the values 2, 3, 4. Repeating the argument of the “longest path”, we immediately also exclude the case  $N = 3$ , since there would be a pair of coincident halflines in  $\mathbb{S}$ , against the multiplicity-one hypothesis, while for  $N = 4$  we have only two possible situations, described at the bottom of the following figure.

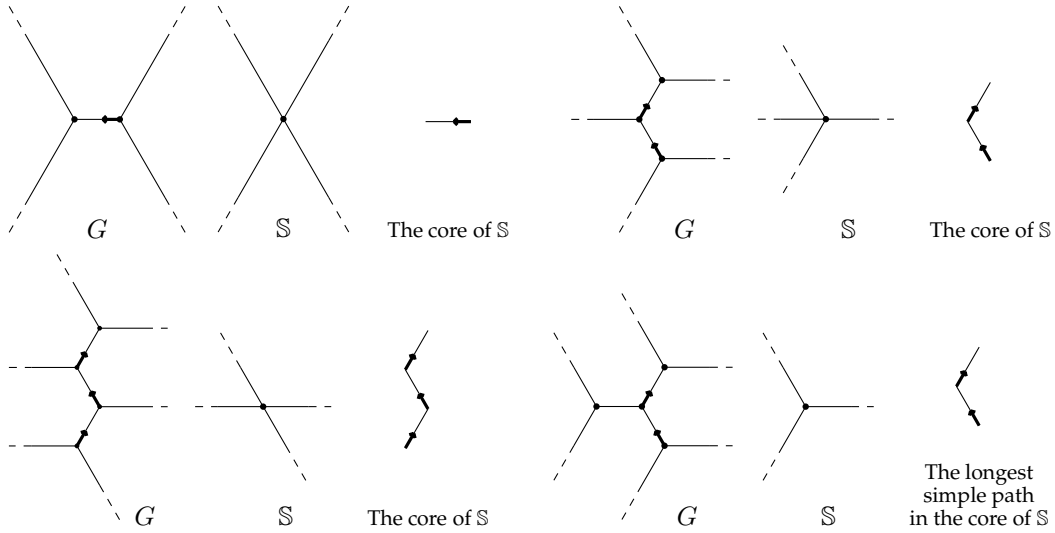


Figure 12: The possible local structure of the graphs  $G$ , with relative networks  $\mathbb{S}$  and cores, for  $N = 2, 3, 4$ .

Hence, if  $N = 4$ , in both two situations above there is in  $\mathbb{S}$  at least one halfline with multiplicity two, thus such case is also excluded.

Then, we conclude that the only possible network with a core is when  $N = 2$  and  $\mathbb{S}$  is given by two lines intersecting at the origin forming angles of 120/60 degrees and the core consists of a collapsed segment which must have an “assigned” unit tangent vector bisecting the two angles of 120 degrees formed by the four halflines.  $\square$

**Lemma 8.10.** *Let  $\mathbb{S} = \bigcup_{i=1}^n \sigma^i(I_i)$  be a degenerate regular shrinker which is  $C_{\text{loc}}^1$ -limit of regular networks homeomorphic to the underlying graph  $G$  of  $\mathbb{S}$  (as in Definition 8.1) and assume that  $\Theta_{\mathbb{S}} < \Theta_{\mathbb{S}^+}$ . Then, the graph  $G$  of  $\mathbb{S}$  is a tree. Thus,  $\mathbb{S}$  is either a multiplicity-one line or a standard triod.*

*Proof.* By the hypotheses, we see that  $G$  is a planar graph. We assume that  $G$  is not a tree, that is, it contains a loop, then we can find a (possibly smaller) loop bounding a region. If such loop is in a core at some point  $P$ , it is easy to see, by the degenerate 120 degrees condition, that such region has six edges and, arguing as in Lemma 8.9, that there must always be at least two non-collapsed, non-straight Abresch–Langer curves arriving/starting at  $P$  in different directions.

Then, if we assume that the complement of  $\mathbb{S}$  in  $\mathbb{R}^2$  contains no bounded components, repeating the argument in the proof of the previous lemma, it follows that  $\mathbb{S}$  consists of a union of halflines for the origin and the loops of  $G$  are all collapsed in the core. Then, by what we said above, there must be at least six halflines emanating from (the core at) the origin. This implies that  $\Theta_{\mathbb{S}} \geq 3$ , which is a contradiction.

Let now  $B$  be a bounded component of the complement of  $\mathbb{S}$  and  $\gamma$  a connected component of the sub-network of  $\mathbb{S}$  which bounds  $B$ , counted with unit multiplicity. Since  $\gamma$  is an embedded, closed curve, smooth with corners and no triple junctions, we can evolve it by “classical” curve shortening

flow  $\gamma_t$ , for  $t \in [-1/2, t_0)$  where we set  $\gamma_{-1/2} = \gamma$ , until it shrinks at some  $t_0 > -1/2$  to a “round” point  $x_0 \in \mathbb{R}^2$  (by the works of Angenent, Gage, Grayson, Hamilton [6–8, 31–33, 35], see Remark 2.2). By the monotonicity formula, we have

$$\int_{\gamma} \rho_{x_0, t_0}(\cdot, -1/2) ds \geq \Theta_{\mathbb{S}^1}$$

and, by the work of Colding–Minicozzi [20, Section 7.2], there holds

$$\Theta_{\mathbb{S}} = \int_{\mathbb{S}} \rho_{0,0}(\cdot, -1/2) d\bar{s} = \sup_{x_0 \in \mathbb{R}^2, t_0 > -1/2} \int_{\mathbb{S}} \rho_{x_0, t_0}(\cdot, -1/2) d\bar{s}.$$

Then,

$$\Theta_{\mathbb{S}} \geq \int_{\mathbb{S}} \rho_{x_0, t_0}(\cdot, -1/2) d\bar{s} \geq \int_{\gamma} \rho_{x_0, t_0}(\cdot, -1/2) ds \geq \Theta_{\mathbb{S}^1},$$

which is a contradiction and we are done.  $\square$

## 8.2 Some geometric properties of the flow

Before proceeding, we show some geometric properties of the curvature flow of a network that we will need in the sequel.

**Proposition 8.11.** *Let  $\mathbb{S}_t$  be the curvature flow of a regular network in a smooth, convex, bounded, open set  $\Omega$ , with fixed end–points on the boundary of  $\Omega$ , for  $t \in [0, T)$ . Then, for every time  $t \in [0, T)$ , the network  $\mathbb{S}_t$  intersects the boundary of  $\Omega$  only at the end–points and such intersections are transversal for every positive time. Moreover,  $\mathbb{S}_t$  remains embedded.*

*Proof.* By continuity, the 3–points cannot hit the boundary of  $\Omega$  at least for some time  $T' > 0$ . The convexity of  $\Omega$  and the strong maximum principle (see [71]) imply that the network cannot intersect the boundary for the first time at an inner regular point. As a consequence, if  $t_0 > 0$  is the “first time” when the  $\mathbb{S}_t$  intersects the boundary at an inner point, this latter has to be a 3–point. The minimality of  $t_0$  is then easily contradicted by the convexity of  $\Omega$ , the 120 degrees condition and the nonzero length of the curves of  $\mathbb{S}_{t_0}$ .

Even if some of the curves of the initial network are tangent to  $\partial\Omega$  at the end–points, by the strong maximum principle, as  $\Omega$  is convex, the intersections become immediately transversal and stay so for every subsequent time.

Finally, if the evolution  $\mathbb{S}_t$  loses embeddedness for the first time, this cannot happen neither at a boundary point, by the argument above, nor at a 3–point, by the 120 degrees condition. Hence it must happen at interior regular points, but this contradicts the strong maximum principle.  $\square$

**Proposition 8.12.** *In the same hypotheses of the previous proposition, if the smooth, bounded, open set  $\Omega$  is strictly convex, for every fixed end–point  $P^r$  on the boundary of  $\Omega$ , for  $r \in \{1, 2, \dots, l\}$ , there is a time  $t_r \in (0, T)$  and an angle  $\alpha_r$  smaller than  $\pi/2$  such that the curve of the network arriving at  $P^r$  form an angle less than  $\alpha_r$  with the inner normal to the boundary of  $\Omega$ , for every time  $t \in (t_r, T)$ .*

*Proof.* We observe that the evolving network  $\mathbb{S}_t$  is contained in the convex set  $\Omega_t \subset \Omega$ , obtained by letting  $\partial\Omega$  (which is a finite set of smooth curves with end–points  $P^r$ ) move by curvature keeping fixed the end–points  $P^r$  (see [43, 80, 81]). By the strict convexity of  $\Omega$  and strong maximum principle, for every positive  $t > 0$ , the two curves of the boundary of  $\Omega$  concurring at  $P^r$  form an angle smaller than  $\pi$  which is not increasing in time. Hence, the statement of the proposition follows.  $\square$

We briefly discuss now the behavior of the area of regions enclosed by the evolving regular network  $\mathbb{S}_t$ . Let us suppose that a (moving) region  $\mathcal{A}(t)$  is bounded by some curves  $\gamma^1, \gamma^2, \dots, \gamma^m$  and let  $A(t)$  its area. Possibly reparametrizing these curves which form the loop  $\ell = \bigcup_{i=1}^m \gamma^i$  in the network, we can assume that  $\ell$  is parametrized counterclockwise, hence, the curvature  $k$  is positive at the convexity points of the boundary of  $\mathcal{A}(t)$ . Then, we have

$$A'(t) = - \sum_{i=1}^m \int_{\gamma^i} \langle k\nu \mid \nu \rangle ds = - \sum_{i=1}^m \int_{\gamma^i} k ds = - \sum_{i=1}^m \Delta\theta_i$$

where  $\Delta\theta_i$  is the difference in the angle between the unit tangent vector  $\tau$  and the unit coordinate vector  $e_1 \in \mathbb{R}^2$  at the final and initial point of the curve  $\gamma^i$ , indeed (supposing the unit tangent vector of the curve  $\gamma^i$  “lives” in the second quadrant of  $\mathbb{R}^2$  – the other cases are analogous) there holds

$$\partial_s \theta_i = \partial_s \arccos(\tau | e_1) = -\frac{\langle \tau_s | e_1 \rangle}{\sqrt{1 - \langle \tau | e_1 \rangle^2}} = k,$$

so

$$A'(t) = -\sum_{i=1}^m \int_{\gamma^i} \partial_s \theta_i ds = -\sum_{i=1}^m \Delta\theta_i$$

Being  $\ell$  a closed loop and considering that at all the end-points of the curves  $\gamma^i$  the angle of the unit tangent vector “jumps” of 120 degrees, we have

$$m\pi/3 + \sum_{i=1}^m \Delta\theta_i = 2\pi,$$

hence,

$$A'(t) = -(2 - m/3)\pi \quad (8.2)$$

(this is called *von Neumann rule*, see [65]).

An immediate consequence is that the area of every region fully bounded by the curves of the network evolves linearly and, more precisely, it increases if the region has more than six edges, it is constant with six edges and it decreases if the edges are less than six. Moreover, this implies that if a region with less than six edges is present, with area  $A_0$  at time  $t = 0$ , the maximal time  $T$  of existence of a smooth flow is finite and

$$T \leq \frac{A_0}{(2 - m/3)\pi} \leq \frac{3A_0}{\pi}.$$

*Remark 8.13.* Since every bounded region contained in a shrinker must decrease its area during the curvature flow of such shrinker (since it is homothetically contracting), another consequence is that the only compact regions that can be present in a regular shrinker are bounded by less than six curves (actually, this conclusion also holds for the “visible” regions – not the cores – of any degenerate regular shrinker).

Moreover, letting a shrinker evolve, since every bounded region must collapse after a time interval of  $1/2$ , the area of such region is only dependent by the number  $m$  of its edges (less than 6), by equation (8.2), indeed

$$A(0) = A(0) - A(1/2) = -\int_0^{1/2} A'(t) dt = \int_0^{1/2} (2 - m/3)\pi dt = (2 - m/3)\pi/2.$$

This implies that the possible structures (topology) of the shrinkers with equibounded diameter are finite.

It is actually conjectured in [40, Conjecture 3.26] that there is an upper bound for the possible number of bounded regions of a shrinker. This would imply that the possible topological structures of shrinkers are finite.

### 8.3 Limits of rescaling procedures

Given a sequence  $\mu_i \nearrow +\infty$  and a space-time point  $(x_0, t_0)$ , where  $0 < t_0 \leq T$ , with  $T$  the maximal time of smooth existence, we consider as before in Section 7.1, the sequence of parabolically rescaled curvature flows  $F_t^{\mu_i}$  in the whole  $\mathbb{R}^2$ , that we denote with  $\mathbb{S}_t^{\mu_i}$ .

We know that, by rescaling the monotonicity formula (end of Section 7.1),

$$\lim_{i \rightarrow \infty} \int_t^0 \int_{\mathbb{S}_s^{\mu_i}} \left| \underline{k} - \frac{x^\perp}{2s} \right|^2 \rho_{0,0}(\cdot, s) ds ds = 0, \quad (8.3)$$

for every  $t \in (-\infty, 0)$ . We see now that this implies that there exists a subsequence of parabolic rescalings which “converges” to a (possibly empty) degenerate, self-similarly shrinking network flow.

**Definition 8.14.** We say that a (possibly degenerate and with multiplicity) network  $\mathbb{S}$  has *bounded length ratios* by the constant  $C > 0$ , if

$$\overline{\mathcal{H}}^1(\mathbb{S} \cap B_R(\overline{x})) \leq CR,$$

for every  $\overline{x} \in \mathbb{R}^2$  and  $R > 0$  ( $\overline{\mathcal{H}}^1$  is the one-dimensional Hausdorff measure counting multiplicities).

Notice that this is a scaling invariant property, with the same constant  $C$ .

**Lemma 8.15.** For any  $\mu > 0$ , let  $\mathbb{S}_t^\mu$  be the parabolically rescaled flow around some  $(x_0, t_0) \in \mathbb{R}^2 \times (0, T)$ , as defined in formula (7.2).

1. There exists a constant  $C = C(\mathbb{S}_0)$  such that, for every  $\overline{x} \in \mathbb{R}^2$ ,  $t \in [0, T)$  and  $R > 0$  there holds

$$\mathcal{H}^1(\mathbb{S}_t \cap B_R(\overline{x})) \leq CR.$$

That is, the family of networks  $\mathbb{S}_t$  has uniformly bounded length ratios by  $C$ . It follows that for every  $\overline{x} \in \mathbb{R}^2$ ,  $t \in [-\mu^2 t_0, 0]$ ,  $\mu > 0$  and  $R > 0$ , we have

$$\mathcal{H}^1(\mathbb{S}_t^\mu \cap B_R(\overline{x})) \leq CR.$$

2. For any  $\varepsilon > 0$  there is a uniform radius  $R = R(\varepsilon)$  such that

$$\int_{\mathbb{S}_t^\mu \setminus B_R(\overline{x})} e^{-|x|^2/2} ds \leq \varepsilon,$$

that is, the family of measures  $e^{-|x|^2/2} \mathcal{H}^1 \llcorner \mathbb{S}_t^\mu$  is tight (see [23]).

*Proof.* By Definition 2.3, if  $\mathbb{S}_0$  is an open network, the number of unbounded curves ( $C^1$ -asymptotic to straight lines) is finite. Then, it is easy to see that, open or not,  $\mathbb{S}_0$  has bounded length ratios, that is, there exists a constant  $C > 0$  such that

$$\mathcal{H}^1(\mathbb{S}_0 \cap B_R(\overline{x})) \leq C'R, \tag{8.4}$$

for all  $\overline{x} \in \mathbb{R}^2$  and  $R > 0$ . This implies that the *entropy* of  $\mathbb{S}_0$  (see [20, 58]) is bounded, that is,

$$E(\mathbb{S}_0) = \sup_{\overline{x} \in \mathbb{R}^2, \tau > 0} \int_{\mathbb{S}_0} \frac{e^{-\frac{|x-\overline{x}|^2}{4\tau}}}{\sqrt{4\pi\tau}} ds = \sup_{\overline{x} \in \mathbb{R}^2, \tau > 0} \int_{\mathbb{S}_0} \rho_{\overline{x}, \tau}(\cdot, 0) ds \leq C''. \tag{8.5}$$

Indeed, for any  $\overline{x} \in \mathbb{R}^2$  and  $\tau > 0$ , changing variable as  $y = (x - \overline{x})/2\tau$ , we have

$$\begin{aligned} \int_{\mathbb{S}_0} \frac{e^{-\frac{|x-\overline{x}|^2}{4\tau}}}{\sqrt{4\pi\tau}} ds &= \int_{\frac{\mathbb{S}_0 - \overline{x}}{2\tau}} \frac{e^{-\frac{|y|^2}{2}}}{\sqrt{2\pi}} ds \\ &= \sum_{n=0}^{\infty} \int_{\frac{\mathbb{S}_0 - \overline{x}}{2\tau} \cap (B_{n+1}(0) \setminus B_n(0))} \frac{e^{-\frac{|y|^2}{2}}}{\sqrt{2\pi}} ds \\ &\leq \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{-n^2/2} \mathcal{H}^1\left(\frac{\mathbb{S}_0 - \overline{x}}{2\tau} \cap B_{n+1}(0)\right) \\ &= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{-n^2/2} \mathcal{H}^1\left(\frac{1}{2\tau} (\mathbb{S}_0 \cap B_{2\tau(n+1)}(\overline{x}) - \overline{x})\right) \\ &= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{-n^2/2} \mathcal{H}^1(\mathbb{S}_0 \cap B_{2\tau(n+1)}(\overline{x})) \frac{1}{2\tau} \\ &\leq \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{-n^2/2} (n+1)C' \\ &= C' \end{aligned}$$



since the series converges (in the last inequality we applied estimate (8.4)).

Then, by the monotonicity formula (7.1), for any  $\bar{x} \in \mathbb{R}^2$ ,  $t \in [0, T)$  and  $R > 0$ , by setting  $\tau = t + R^2$ , we have

$$\int_{\mathbb{S}_t} \frac{e^{-\frac{|x-\bar{x}|^2}{4R^2}}}{\sqrt{4\pi}R} ds = \int_{\mathbb{S}_t} \rho_{\bar{x}, t+R^2}(\cdot, t) ds \leq \int_{\mathbb{S}_0} \rho_{\bar{x}, t+R^2}(\cdot, 0) ds \leq C'',$$

hence,

$$\mathcal{H}^1(\mathbb{S}_t \cap B_R(\bar{x})) \leq \sqrt{4\pi}eR \int_{\mathbb{S}_t \cap B_R(\bar{x})} \frac{e^{-\frac{|x-\bar{x}|^2}{4R^2}}}{\sqrt{4\pi}R} ds \leq \sqrt{4\pi}C''eR.$$

Since this conclusion is scaling invariant, it also holds for all the rescaled networks  $\mathbb{S}_t^{\mu_i}$  and the first point of the lemma follows with  $C = \sqrt{4\pi}C''e$ . The second point is a consequence of the first one, indeed, we have

$$\begin{aligned} \int_{\mathbb{S}_t^{\mu_i} \setminus B_R(\bar{x})} e^{-\frac{|x|^2}{2}} ds &= \sum_{n=1}^{\infty} \int_{\mathbb{S}_t^{\mu_i} \cap (B_{(n+1)R}(\bar{x}) \setminus B_{nR}(\bar{x}))} e^{-\frac{|x|^2}{2}} ds \\ &\leq \sum_{n=1}^{\infty} e^{-n^2 R^2/2} \mathcal{H}^1(\mathbb{S}_t^{\mu_i} \cap B_{(n+1)R}(\bar{x})) \\ &\leq C \sum_{n=1}^{\infty} e^{-n^2 R^2/2} (n+1)R \\ &= f(R) \end{aligned}$$

and the function  $f$  satisfies  $\lim_{R \rightarrow +\infty} f(R) = 0$ .  $\square$

**Proposition 8.16.** *Given a sequence of parabolically rescaled curvature flows  $\mathbb{S}_t^{\mu_i}$ , as above, there exists a subsequence  $\mu_{i_j}$  and a (possibly empty) degenerate regular self-similarly shrinking network flow  $\mathbb{S}_t^\infty$  such that for almost all  $t \in (-\infty, 0)$  and for any  $\alpha \in (0, 1/2)$ ,*

$$\mathbb{S}_t^{\mu_{i_j}} \rightarrow \mathbb{S}_t^\infty$$

in  $C_{\text{loc}}^{1,\alpha} \cap W_{\text{loc}}^{2,2}$ . This convergence also holds in the sense of Radon measures for all  $t \in (-\infty, 0)$ .

Moreover, for every continuous function with compact support in space-time  $\varphi : \mathbb{R}^2 \times (-\infty, 0) \rightarrow \mathbb{R}$  there holds

$$\lim_{j \rightarrow \infty} \int_{(-\infty, 0)} \int_{\mathbb{S}_t^{\mu_{i_j}}} \varphi(\cdot, t) ds dt = \int_{(-\infty, 0)} \int_{\mathbb{S}_t^\infty} \varphi(\cdot, t) d\bar{s} dt, \quad (8.6)$$

where  $d\bar{s}$  denotes the integration with respect to the canonical measure on  $\mathbb{S}_t^\infty$ , counting multiplicities, and

$$\lim_{j \rightarrow \infty} \int_{\mathbb{S}_t^{\mu_{i_j}}} \rho_{0,0}(\cdot, t) ds = \int_{\mathbb{S}_t^\infty} \rho_{0,0}(\cdot, t) d\bar{s} = \Theta_{\mathbb{S}_{-1/2}^\infty} = \hat{\Theta}(x_0, t_0), \quad (8.7)$$

for every  $t \in (-\infty, 0)$ .

*Proof.* We follow ideas in Ilmanen [46, Lemma 8] and [45, Section 7.1].

By the first point of Lemma 8.15, for every ball  $B_R$  centered at the origin of  $\mathbb{R}^2$ , we have the uniform bound  $\mathcal{H}^1(\mathbb{S}_t^{\mu_i} \cap B_R) \leq CR$ , for some constant  $C$  independent of  $i \in \mathbb{N}$  and  $t \in (-\infty, 0)$ . Hence, we can assume that the sequence of Radon measures defined by the left side of equation (8.6) are locally equibounded and converges to some limit measure in the space-time ambient  $\mathbb{R}^2 \times (-\infty, 0)$

Considering the functions

$$f_i(t) = \int_{\mathbb{S}_t^{\mu_i}} \left| \underline{k} - \frac{x^\perp}{2t} \right|^2 \rho_{0,0}(\cdot, t) ds,$$

the limit (8.3) implies that  $f_i \rightarrow 0$  in  $L_{\text{loc}}^1(-\infty, 0)$ . Thus, there exists a (not relabeled) subsequence such that the sequence of functions  $f_i$  converges pointwise almost everywhere to zero. We call  $A \subset (-\infty, 0)$

such a convergence set.

Then, for any  $t \in A$ , because of the uniform bound  $\mathcal{H}^1(\mathbb{S}_t^{\mu_i} \cap B_R) \leq CR$ , we have that for any  $R > 0$

$$\int_{\mathbb{S}_t^{\mu_i} \cap B_R} k^2 ds \leq C_R(t),$$

for a constant  $C_R(t)$  independent of  $i$ . Hence, if  $t \in A$ , reparametrizing the curves of the rescaled networks by arclength, we obtain curves in  $W_{\text{loc}}^{2,2}$  with uniformly bounded first derivatives, which implies that any subsequence of the networks  $\mathbb{S}_t^{\mu_i}$  admits a further subsequence converging weakly in  $W_{\text{loc}}^{2,2}$ , hence in  $C_{\text{loc}}^{1,\alpha}$  to a degenerate regular network  $\mathbb{S}_t^\infty$ . Moreover, such subsequence  $\mathbb{S}_t^{\mu_{i_j}}$  actually converges strongly in  $W_{\text{loc}}^{2,2}$  by the weak convergence in  $W_{\text{loc}}^{2,2}$  and the fact that  $f_i(t) \rightarrow 0$  in  $L_{\text{loc}}^1$ . Finally, by the convergence in  $C_{\text{loc}}^{1,\alpha}$ , the associated Radon measures  $\lambda_t^{i_j} = \mathcal{H}^1 \llcorner \mathbb{S}_t^{\mu_{i_j}}$  weakly converge to  $\lambda_t^\infty = \overline{\mathcal{H}}^1 \llcorner \mathbb{S}_t^\infty$  (where  $\overline{\mathcal{H}}^1 \llcorner \mathbb{S}_t^\infty$  is the one-dimensional Hausdorff measure restricted to  $\mathbb{S}_t^\infty$ , counting multiplicities). Since the integral functional

$$\mathbb{S} \mapsto \int_{\mathbb{S}} \left| \underline{k} - \frac{x^\perp}{2t} \right|^2 \rho_{0,0}(\cdot, t) ds$$

is lower semicontinuous with respect to this convergence (see [77], for instance), the limit  $\mathbb{S}_t^\infty$  satisfies

$$\underline{k} - \frac{x^\perp}{2t} = 0,$$

in  $W_{\text{loc}}^{2,2}$ , hence, by a bootstrap argument, each non-degenerate curve of  $\mathbb{S}_t^\infty$  is actually smooth. Thus, for every  $t \in A$  the network  $\mathbb{S}_t^\infty$  is a degenerate regular shrinker, up to a dilation factor.

By a standard diagonal argument we can assume that for  $t$  in a dense countable subset  $B_1 \subset A$  the subsequence  $\mathbb{S}_t^{\mu_{i_j}}$  converges in  $W_{\text{loc}}^{2,2}$  and  $C_{\text{loc}}^{1,\alpha}$  to a limit degenerate regular shrinker  $\mathbb{S}_t^\infty$ , with associated Radon measure  $\lambda_t^\infty = \overline{\mathcal{H}}^1 \llcorner \mathbb{S}_t^\infty$ , as above.

When  $t \in A \setminus B_1$  we consider as  $\mathbb{S}_t^\infty$  the limit degenerate regular shrinker of an arbitrary converging subsequence of the networks  $\mathbb{S}_t^{\mu_{i_j}}$ , and  $\lambda_t^\infty = \overline{\mathcal{H}}^1 \llcorner \mathbb{S}_t^\infty$ .

When  $t \in (-\infty, 0) \setminus A$  we instead consider as  $\lambda_t^\infty$  the limit Radon measure of an arbitrary weakly-converging subsequence of the Radon measures  $\lambda_t^{i_j} = \mathcal{H}^1 \llcorner \mathbb{S}_t^{\mu_{i_j}}$ .

In this way we defined the limit network  $\mathbb{S}_t^\infty$  for every  $t \in A$  and the limit Radon measures  $\lambda_t^\infty$  for every  $t \in (-\infty, 0)$ .

If  $\mathcal{F}$  is a countable dense family of smooth functions in the cone of non negative functions in  $C_c^0(\mathbb{R}^2)$ , by the above convergence and the rescaled monotonicity formula, it follows that for every  $\varphi \in \mathcal{F}$ , there holds (by Proposition 6.14 and formula (6.2))

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{S}_t^{\mu_{i_j}}} \varphi ds &= - \int_{\mathbb{S}_t^{\mu_{i_j}}} \varphi k^2 ds + \int_{\mathbb{S}_t^{\mu_{i_j}}} \langle \nabla \varphi | \underline{k} \rangle ds \\ &= - \int_{\mathbb{S}_t^{\mu_{i_j}}} \varphi \left| \underline{k} - \frac{\nabla \varphi}{2\varphi} \right|^2 ds + \int_{\mathbb{S}_t^{\mu_{i_j}}} \frac{|\nabla \varphi|^2}{4\varphi} ds \\ &\leq \frac{1}{4} \int_{\mathbb{S}_t^{\mu_{i_j}}} \frac{|\nabla \varphi|^2}{\varphi} ds \\ &\leq (\max |\nabla^2 \varphi|/2) \lambda_t^{i_j}(\{\varphi > 0\}) \\ &\leq C(\varphi, \nabla^2 \varphi), \end{aligned}$$

where we used the estimate  $|\nabla \varphi|^2/\varphi \leq 2 \max |\nabla^2 \varphi|$ , holding for every  $\varphi \in C_c^2(\mathbb{R}^n)$  (where  $\varphi > 0$ ), proved in [45, Lemma 6.6] and the uniform bound  $\mathcal{H}^1(\mathbb{S}_t^{\mu_i} \cap B_R) \leq CR$ , for some constant  $C$  independent of  $i \in \mathbb{N}$  and  $t \in (-\infty, 0)$ .

Hence, fixing a single  $t_0 \in (-\infty, 0) \setminus B_1$ , the function

$$\int_{\mathbb{S}_t^{\mu_{i_j}}} \varphi ds - C(\varphi, \nabla^2 \varphi)t$$

is monotone non increasing once restricted to  $B_1 \cup \{t_0\}$ . Passing to the limit (on the  $t_0$ -special subsequence such that  $\lambda_{t_0}^{i_j}$  converges to  $\lambda_{t_0}^\infty$ ) the same holds for the function

$$t \mapsto \int_{\mathbb{R}^2} \varphi d\lambda_t^\infty - C(\varphi, \nabla^2 \varphi)t,$$

restricted to  $B_1 \cup \{t_0\}$ . By the arbitrariness of  $t_0 \in (-\infty, 0) \setminus B_1$ , we then conclude that such function is monotone non increasing on the whole  $(-\infty, 0)$ . Thus, for every  $\varphi \in \mathcal{F}$  the function  $t \mapsto \int_{\mathbb{R}^2} \varphi d\lambda_t^\infty$  has an at most countable set of (jump) discontinuities, that we call  $B_\varphi$ . Hence, we have that outside a countable subset  $B = \bigcup_{\varphi \in \mathcal{F}} B_\varphi$  of  $(-\infty, 0)$ , all the functions

$$t \mapsto \int_{\mathbb{R}^2} \varphi d\lambda_t^\infty$$

are continuous, for every  $\varphi \in \mathcal{F}$ . This clearly implies that if  $t \in (-\infty, 0) \setminus B$ , then the value of the integral  $\int_{\mathbb{R}^2} \varphi d\lambda_t^\infty$  is uniquely determined and independent of the  $t$ -subsequence chosen to define  $\lambda_t^\infty$ , for every  $\varphi \in \mathcal{F}$ . An immediate consequence is that (by the density of  $\mathcal{F}$ ),

- if  $t \in (-\infty, 0) \setminus B$ , the Radon measure  $\lambda_t^\infty$  is uniquely determined and the full sequence  $\lambda_t^{i_j}$  converges to  $\lambda_t^\infty$ ,
- if  $t \in A$ , the network  $\mathbb{S}_t^\infty$  is uniquely determined and the full sequence  $\mathbb{S}_t^{\mu_{i_j}}$  converges to  $\mathbb{S}_t^\infty$  in  $W_{\text{loc}}^{2,2}$  and  $C_{\text{loc}}^{1,\alpha}$ ,

as  $j \rightarrow \infty$ .

Then, we can conclude by a diagonal argument on the sequences of networks  $\mathbb{S}_t^{\mu_{i_j}}$  when  $t \in B$ , that we have a subsequence (not relabeled) of  $\mu_{i_j}$  such that for every  $t \in A$  the networks  $\mathbb{S}_t^{\mu_{i_j}}$  converge in  $W_{\text{loc}}^{2,2}$  and  $C_{\text{loc}}^{1,\alpha}$  and as Radon measures to  $\mathbb{S}_t^\infty$ , as  $j \rightarrow \infty$ , and for every  $t \in (-\infty, 0)$  we have  $\lambda_t^{i_j} \rightarrow \lambda_t^\infty$  as Radon measures.

By Proposition 6.14, every rescaled flow is a regular Brakke flow with equality, hence, the integrated version of equation (6.2) holds, that is,

$$\int_{\mathbb{R}^2} \varphi(\cdot, t_1) d\lambda_{t_1}^{i_j} - \int_{\mathbb{R}^2} \varphi(\cdot, t_2) d\lambda_{t_2}^{i_j} = \int_{t_2}^{t_1} \left[ - \int_{\mathbb{S}_t^{\mu_{i_j}}} \varphi(\gamma, t) k^2 ds + \int_{\mathbb{S}_t^{\mu_{i_j}}} \langle \nabla \varphi(\gamma, t) | \underline{k} \rangle ds + \int_{\mathbb{S}_t^{\mu_{i_j}}} \varphi_t(\gamma, t) ds \right] dt,$$

for every smooth function with compact support  $\varphi : \mathbb{R}^2 \times (-\infty, 0) \rightarrow \mathbb{R}$  and  $t_1, t_2 \in (-\infty, 0)$ .

By the  $W_{\text{loc}}^{2,2}$ -convergence almost everywhere (for  $t$  in the set  $A$ ) and the limit (8.3) (which allows us to use the dominated convergence theorem) we can pass to the limit to get

$$\int_{\mathbb{R}^2} \varphi(\cdot, t_1) d\lambda_{t_1}^\infty - \int_{\mathbb{R}^2} \varphi(\cdot, t_2) d\lambda_{t_2}^\infty = \int_{t_2}^{t_1} \left[ - \int_{\mathbb{S}_t^\infty} \varphi(\gamma, t) k^2 d\bar{s} + \int_{\mathbb{S}_t^\infty} \langle \nabla \varphi(\gamma, t) | \underline{k} \rangle d\bar{s} + \int_{\mathbb{S}_t^\infty} \varphi_t(\gamma, t) d\bar{s} \right] dt,$$

where  $d\bar{s}$  denotes the integration with respect to the canonical measure on  $\mathbb{S}_t^\infty$ , counting multiplicities. This shows that the function  $t \mapsto \int_{\mathbb{R}^2} \varphi(\cdot, t) d\lambda_t^\infty$  is absolutely continuous on  $(-\infty, 0)$  and for almost every  $t \in (-\infty, 0)$ , there holds

$$\frac{d}{dt} \int_{\mathbb{R}^2} \varphi(\cdot, t) d\lambda_t^\infty = - \int_{\mathbb{S}_t^\infty} \varphi(\gamma, t) k^2 d\bar{s} + \int_{\mathbb{S}_t^\infty} \langle \nabla \varphi(\gamma, t) | \underline{k} \rangle d\bar{s} + \int_{\mathbb{S}_t^\infty} \varphi_t(\gamma, t) d\bar{s}. \quad (8.8)$$

We then consider, for every  $t \in (-\infty, 0)$ , the Radon measures defined by

$$\nu_t(D) = \lambda_t^\infty(\sqrt{-2t} D) / \sqrt{-2t}.$$

It is easy to see that showing that  $\lambda_t^\infty = \overline{\mathcal{H}}^1 \llcorner (\sqrt{-2t} \mathbb{S}_{-1/2}^\infty)$  for every  $t \in (-\infty, 0)$ , is equivalent to prove that the measures  $\nu_t$  are all the same and this means that  $\mathbb{S}_t^\infty$  is a degenerate regular self-similarly shrinking network flow.

We have, for every smooth function with compact support  $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,

$$\int_{\mathbb{R}^2} \psi(x) d\nu_t(x) = \frac{1}{\sqrt{-2t}} \int_{\mathbb{R}^2} \psi\left(\frac{x}{\sqrt{-2t}}\right) d\lambda_t^\infty(x),$$

hence, choosing  $\varphi(x, t) = \psi\left(\frac{x}{\sqrt{-2t}}\right)$ , at every time  $t$  such that equality (8.8) holds (almost every  $t \in (-\infty, 0)$ ), we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} \psi(x) d\nu_t(x) &= \frac{1}{-2t\sqrt{-2t}} \int_{\mathbb{S}_t^\infty} \psi\left(\frac{\gamma}{\sqrt{-2t}}\right) d\bar{s} - \frac{1}{\sqrt{-2t}} \int_{\mathbb{S}_t^\infty} \psi\left(\frac{\gamma}{\sqrt{-2t}}\right) k^2 d\bar{s} \\ &\quad + \frac{1}{-2t} \int_{\mathbb{S}_t^\infty} \left\langle \nabla \psi\left(\frac{\gamma}{\sqrt{-2t}}\right) \middle| \underline{k} \right\rangle d\bar{s} + \int_{\mathbb{S}_t^\infty} \left\langle \nabla \psi\left(\frac{\gamma}{\sqrt{-2t}}\right) \middle| \frac{\gamma}{4t^2} \right\rangle d\bar{s}. \end{aligned}$$

Substituting  $\underline{k} = \gamma^\perp / 2t$ , we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} \psi(x) d\nu_t(x) &= \frac{1}{-2t\sqrt{-2t}} \int_{\mathbb{S}_t^\infty} \psi\left(\frac{\gamma}{\sqrt{-2t}}\right) d\bar{s} - \frac{1}{\sqrt{-2t}} \int_{\mathbb{S}_t^\infty} \psi\left(\frac{\gamma}{\sqrt{-2t}}\right) \frac{\langle \underline{k} | \gamma^\perp \rangle}{2t} d\bar{s} \\ &\quad - \int_{\mathbb{S}_t^\infty} \left\langle \nabla \psi\left(\frac{\gamma}{\sqrt{-2t}}\right) \middle| \frac{\gamma^\perp}{4t^2} \right\rangle d\bar{s} + \int_{\mathbb{S}_t^\infty} \left\langle \nabla \psi\left(\frac{\gamma}{\sqrt{-2t}}\right) \middle| \frac{\gamma}{4t^2} \right\rangle d\bar{s} \\ &= \frac{1}{-2t\sqrt{-2t}} \int_{\mathbb{S}_t^\infty} \psi\left(\frac{\gamma}{\sqrt{-2t}}\right) d\bar{s} - \frac{1}{\sqrt{-2t}} \int_{\mathbb{S}_t^\infty} \psi\left(\frac{\gamma}{\sqrt{-2t}}\right) \frac{\langle \underline{k} | \gamma^\perp \rangle}{2t} d\bar{s} \\ &\quad + \int_{\mathbb{S}_t^\infty} \left\langle \nabla \psi\left(\frac{\gamma}{\sqrt{-2t}}\right) \middle| \frac{\gamma^\top}{4t^2} \right\rangle d\bar{s} \\ &= \frac{1}{-2t\sqrt{-2t}} \int_{\mathbb{S}_t^\infty} \left[ \psi\left(\frac{\gamma}{\sqrt{-2t}}\right) + \psi\left(\frac{\gamma}{\sqrt{-2t}}\right) \langle \underline{k} | \gamma \rangle + \left\langle \nabla \psi\left(\frac{\gamma}{\sqrt{-2t}}\right) \middle| \frac{\tau}{\sqrt{-2t}} \right\rangle \langle \tau | \gamma \rangle \right] d\bar{s}, \end{aligned}$$

where we denoted with  $\gamma^\top$  the tangential component of the vector  $\gamma \in \mathbb{R}^2$ , that is,  $\gamma^\top = \langle \tau | \gamma \rangle \tau$ . Noticing now that

$$\begin{aligned} \partial_s \left[ \psi\left(\frac{\gamma}{\sqrt{-2t}}\right) \langle \tau | \gamma \rangle \right] &= \left\langle \nabla \psi\left(\frac{\gamma}{\sqrt{-2t}}\right) \middle| \frac{\tau}{\sqrt{-2t}} \right\rangle \langle \tau | \gamma \rangle + \psi\left(\frac{\gamma}{\sqrt{-2t}}\right) \langle \underline{k} | \gamma \rangle + \psi\left(\frac{\gamma}{\sqrt{-2t}}\right) \langle \tau | \tau \rangle \\ &= \left\langle \nabla \psi\left(\frac{\gamma}{\sqrt{-2t}}\right) \middle| \frac{\tau}{\sqrt{-2t}} \right\rangle \langle \tau | \gamma \rangle + \psi\left(\frac{\gamma}{\sqrt{-2t}}\right) \langle \underline{k} | \gamma \rangle + \psi\left(\frac{\gamma}{\sqrt{-2t}}\right), \end{aligned}$$

we conclude

$$\frac{d}{dt} \int_{\mathbb{R}^2} \psi(x) d\nu_t(x) = \frac{1}{-2t\sqrt{-2t}} \int_{\mathbb{S}_t^\infty} \partial_s \left[ \psi\left(\frac{\gamma}{\sqrt{-2t}}\right) \langle \tau | \gamma \rangle \right] d\bar{s}$$

and this last integral is zero by Lemma 8.3 and the last point of Remark 8.5.

Since for every map  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$  the function  $t \mapsto \int_{\mathbb{R}^2} \varphi(x) d\nu_t(x)$  is absolutely continuous on  $(-\infty, 0)$  with zero derivative almost everywhere, it is constant and we are done.

Equation (8.6) clearly follows by the convergence assumption on the sequence of Radon measures in  $\mathbb{R}^2 \times (-\infty, 0)$  and this conclusion.

Finally, for every  $t \in (-\infty, 0)$ , by the second point of Lemma 8.15, we can pass to the limit in the Gaussian integral and we get

$$\lim_{j \rightarrow \infty} \int_{\mathbb{S}_t^{\mu_{i_j}}} \rho_{0,0}(\cdot, t) ds = \int_{\mathbb{S}_t^\infty} \rho_{0,0}(\cdot, t) d\bar{s} = \Theta_{\mathbb{S}_{-1/2}^\infty},$$

since the right integral is constant in  $t$ , being  $\mathbb{S}_t^\infty$  a self-similarly shrinking flow. Recalling that (see Section 7.1)

$$\int_{\mathbb{S}_t^{\mu_{i_j}}} \rho_{0,0}(\cdot, t) ds = \Theta_{x_0, t_0}(t_0 + \mu_{i_j}^{-2} t) \rightarrow \widehat{\Theta}(x_0, t_0),$$

as  $j \rightarrow \infty$ , equality (8.7) follows.  $\square$

*Remark 8.17.* We underline that even if the limit flow is composed of homothetic rescalings of a single degenerate regular network, we cannot conclude that the convergence of  $\mathbb{S}_t^{\mu_{ij}}$  to  $\mathbb{S}_t^\infty$  is in  $W_{\text{loc}}^{2,2}$  and  $C_{\text{loc}}^{1,\alpha}$  for every  $t \in (-\infty, 0)$ , but only for almost every  $t \in (-\infty, 0)$ . For the “other” times, the convergence could be only as Radon measures.

The following lemma is helpful in strengthening the convergence in the previous proposition.

**Lemma 8.18.** *Given a sequence of smooth curvature flows of networks  $\mathbb{S}_t^i$  in a time interval  $(t_1, t_2)$  with uniformly bounded length ratios, if in a dense subset of times  $t \in (t_1, t_2)$  the networks  $\mathbb{S}_t^i$  converge in a ball  $B \subset \mathbb{R}^2$  in  $C_{\text{loc}}^1$ , as  $i \rightarrow \infty$ , to a multiplicity-one, embedded,  $C^\infty$ -curve  $\gamma_t$  moving by curvature in  $B' \supset \overline{B}$ , for  $t \in (t_1, t_2]$  (hence, the curvature of  $\gamma_t$  is uniformly bounded), then for every  $(x_0, t_0) \in B \times (t_1, t_2]$ , the curvature of  $\mathbb{S}_t^i$  is uniformly bounded in a neighborhood of  $(x_0, t_0)$  in space-time. It follows that, for every  $(x_0, t_0) \in B \times (t_1, t_2]$ , we have  $\mathbb{S}_t^i \rightarrow \gamma_t$  smoothly around  $(x_0, t_0)$  in space-time (possibly, up to local reparametrizations of the networks  $\mathbb{S}_t^i$ ).*

*Proof.* Being  $\gamma_t$  a smooth flow of an embedded curve in  $B$ , for  $(x, t)$  in a suitably small neighborhood of  $(x_0, t_0) \in B \times (t_1, t_2]$  we have that  $\Theta_{x,t}(\tau) \leq 1 + \varepsilon/2 < 3/2$ , for every  $\tau \in (\tau_0, t)$  and some  $\tau_0 > 0$ , where  $\varepsilon > 0$  is smaller than the “universal” constant given by White’s local regularity theorem in [84]. Then, in a possibly smaller space-time neighborhood of  $(x_0, t_0)$ , for a fixed time  $\bar{\tau} \in (\tau_0, t)$  where the  $C_{\text{loc}}^1$ -convergence of the networks  $\mathbb{S}_{\bar{\tau}}^i \rightarrow \gamma_{\bar{\tau}}$  holds (such a subset of times is dense), for  $i$  large enough, the Gaussian density functions of  $\mathbb{S}_{\bar{\tau}}^i$  satisfy  $\Theta_{x,\bar{\tau}}^i(\bar{\tau}) < 1 + \varepsilon < 3/2$  (the Gaussian density functions are clearly continuous under the  $C_{\text{loc}}^1$  convergence with uniform length ratios estimate, by the exponential decay of backward heat kernel). Hence, by the monotonicity formula this also holds for every  $\tau \in (\bar{\tau}, t)$ . In other words,  $\Theta_{x,t}^i(t - r^2) < 1 + \varepsilon < 3/2$ , for every  $(x, t)$  in a space-time neighborhood of  $(x_0, t_0)$ ,  $0 < r < r_0$  and  $i > i_0$ , for some  $r_0 > 0$ . Notice that this “forbids” the presence of a 3-point of  $\mathbb{S}_t^i$  in such space-time neighborhood.

Then, White’s theorem (see Theorem 3.5 in [84]) gives a uniform local (in space-time) estimate on the curvature of all  $\mathbb{S}_t^i$ , which actually implies uniform bounds on all its higher derivatives (for instance, by Ecker and Huisken interior estimates in [26]), around  $(x_0, t_0)$ . Hence, the statement of the lemma follows.  $\square$

As a consequence, the convergence of  $\mathbb{S}_t^{\mu_{ij}}$  to the limit degenerate regular self-similarly shrinking network flow  $\mathbb{S}_t^\infty$ , in Proposition 8.16, is smooth locally in space-time around every interior point of the multiplicity-one curves of the network  $\mathbb{S}_t^\infty$ .

Moreover, if  $\mathbb{S}_t^\infty$  is non-degenerate (no cores) and with only multiplicity-one curves, then actually  $\mathbb{S}_t^{\mu_{ij}} \rightarrow \mathbb{S}_t^\infty$  smoothly, locally in space-time (also around the 3-points). This can be shown by following the argument of the proof of Lemma 8.6 in [47] (see anyway the proof in the special case of Lemma 9.1).

We deal now with the possible blow-up limits arising from Huisken’s dynamical procedure. We recall that

$$\tilde{\rho}(x) = e^{-\frac{|x|^2}{2}}.$$

The following technical lemma is the exact analogue of Lemma 8.15 for Huisken’s rescaling procedure. It follows in the same way by the first point of such lemma.

**Lemma 8.19.** *Let  $\tilde{\mathbb{S}}_{x_0,t}$  be the family of rescaled networks, obtained via Huisken’s dynamical procedure around some  $x_0 \in \mathbb{R}^2$ , as defined in formula (7.3).*

1. *There exists a constant  $C = C(\mathbb{S}_0)$  such that, for every  $\bar{x}, x_0 \in \mathbb{R}^2$ ,  $t \in [-\frac{1}{2} \log T, +\infty)$  and  $R > 0$  there holds*

$$\mathcal{H}^1(\tilde{\mathbb{S}}_{x_0,t} \cap B_R(\bar{x})) \leq CR.$$

2. *For any  $\varepsilon > 0$  there is a uniform radius  $R = R(\varepsilon)$  such that*

$$\int_{\tilde{\mathbb{S}}_{x_0,t} \setminus B_R(\bar{x})} e^{-|x|^2/2} ds \leq \varepsilon,$$

*that is, the family of measures  $e^{-|x|^2/2} \mathcal{H}^1 \llcorner \tilde{\mathbb{S}}_{x_0,t}$  is tight (see [23]).*

**Proposition 8.20.** *Let  $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i([0, 1], t)$  be a  $C^{2,1}$  curvature flow of regular networks in the time interval  $[0, T]$ , then, for every  $x_0 \in \mathbb{R}^2$  and for every subset  $\mathcal{I}$  of  $[-1/2 \log T, +\infty)$  with infinite Lebesgue measure, there exists a sequence of rescaled times  $t_j \rightarrow +\infty$ , with  $t_j \in \mathcal{I}$ , such that the sequence of rescaled networks  $\tilde{\mathbb{S}}_{x_0, t_j}$  (obtained via Huisken's dynamical procedure) converges in  $C_{\text{loc}}^{1,\alpha} \cap W_{\text{loc}}^{2,2}$ , for any  $\alpha \in (0, 1/2)$ , to a (possibly empty) limit network, which is a degenerate regular shrinker  $\tilde{\mathbb{S}}_\infty$  (possibly with multiplicity). Moreover, we have*

$$\lim_{j \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{\tilde{\mathbb{S}}_{x_0, t_j}} \tilde{\rho} d\sigma = \frac{1}{\sqrt{2\pi}} \int_{\tilde{\mathbb{S}}_\infty} \tilde{\rho} d\bar{\sigma} = \Theta_{\tilde{\mathbb{S}}_\infty} = \hat{\Theta}(x_0). \quad (8.9)$$

where  $d\bar{\sigma}$  denotes the integration with respect to the canonical measure on  $\tilde{\mathbb{S}}_\infty$ , counting multiplicities.

*Proof.* Letting  $t_1 = -1/2 \log T$  and  $t_2 \rightarrow +\infty$  in the rescaled monotonicity formula (7.4), by Lemma 7.6 we get

$$\int_{-1/2 \log T}^{+\infty} \int_{\tilde{\mathbb{S}}_{x_0, t}} |\tilde{k} + x^\perp|^2 \tilde{\rho} d\sigma dt < +\infty,$$

which implies

$$\int_{\mathcal{I}} \int_{\tilde{\mathbb{S}}_{x_0, t}} |\tilde{k} + x^\perp|^2 \tilde{\rho} d\sigma dt < +\infty.$$

Being the last integral finite and being the integrand a non negative function on a set of infinite Lebesgue measure, we can extract within  $\mathcal{I}$  a sequence of times  $t_j \rightarrow +\infty$ , such that

$$\lim_{j \rightarrow +\infty} \int_{\tilde{\mathbb{S}}_{x_0, t_j}} |\tilde{k} + x^\perp|^2 \tilde{\rho} d\sigma = 0. \quad (8.10)$$

It follows that, for every ball  $B_R$  of radius  $R$  in  $\mathbb{R}^2$ , the networks  $\tilde{\mathbb{S}}_{x_0, t_j}$  have curvature uniformly bounded in  $L^2(B_R)$ . Moreover, by the first point of Lemma 8.19, for every ball  $B_R$  centered at the origin of  $\mathbb{R}^2$  we have the uniform bound  $\mathcal{H}^1(\tilde{\mathbb{S}}_{x_0, t_j} \cap B_R) \leq CR$ , for some constant  $C$  independent of  $j \in \mathbb{N}$ . Then, reparametrizing the rescaled networks by arclength, we obtain curves with uniformly bounded first derivatives and with second derivatives in  $L_{\text{loc}}^2$ .

By a standard compactness argument (see [42, 53]), the sequence  $\tilde{\mathbb{S}}_{x_0, t_j}$  of reparametrized networks admits a subsequence  $\tilde{\mathbb{S}}_{x_0, t_{j_l}}$  which converges, weakly in  $W_{\text{loc}}^{2,2}$  and strongly in  $C_{\text{loc}}^{1,\alpha}$ , to a (possibly empty) limit regular degenerate  $C^1$  network  $\tilde{\mathbb{S}}_\infty$  (possibly with multiplicity). Since the integral functional

$$\tilde{\mathbb{S}} \mapsto \int_{\tilde{\mathbb{S}}} |\tilde{k} + x^\perp|^2 \tilde{\rho} d\sigma$$

is lower semicontinuous with respect to this convergence (see [77], for instance), the limit  $\tilde{\mathbb{S}}_\infty$  satisfies  $\tilde{k}_\infty + x^\perp = 0$  in the sense of distributions.

A priori, the limit network is composed by curves in  $W_{\text{loc}}^{2,2}$ , but from the relation  $\tilde{k}_\infty + x^\perp = 0$ , it follows that the curvature  $\tilde{k}_\infty$  is continuous. By a bootstrap argument, it is then easy to see that  $\tilde{\mathbb{S}}_\infty$  is actually composed by  $C^\infty$  curves.

By means of the second point of Lemma 8.19, we can pass to the limit in the Gaussian integral and we get

$$\lim_{j \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{\tilde{\mathbb{S}}_{x_0, t_j}} \tilde{\rho} d\sigma = \frac{1}{\sqrt{2\pi}} \int_{\tilde{\mathbb{S}}_\infty} \tilde{\rho} d\bar{\sigma} = \Theta_{\tilde{\mathbb{S}}_\infty}.$$

Recalling that

$$\frac{1}{\sqrt{2\pi}} \int_{\tilde{\mathbb{S}}_{x_0, t_j}} \tilde{\rho} d\sigma = \int_{\mathbb{S}_{t(t_j)}} \rho_{x_0}(\cdot, t(t_j)) ds = \Theta_{x_0}(t(t_j)) \rightarrow \hat{\Theta}(x_0)$$

as  $j \rightarrow \infty$ , equality (8.9) follows.

The convergence in  $W_{\text{loc}}^{2,2}$  is implied by the weak convergence in  $W_{\text{loc}}^{2,2}$  and equation (8.10).  $\square$



*Remark 8.21.*

1. In the case of a special rate of blow-up of the curvature, the so called *Type I* singularities, that is, when there exists a constant  $C$  such that

$$\max_{\mathbb{S}_t} k^2 \leq \frac{C}{T-t} \quad (8.11)$$

for every  $t \in [0, T)$ , the proof of this proposition gets easier and we get a stronger convergence to the limit network. This is due to the uniform pointwise bound on the curvature (consequently on its derivatives) that we get after the rescaling by the Type I estimate, see [63, Section 6, Proposition 6.16], for instance. Similarly, with the right choice of the sequence  $\mu_{i_j}$ , the same holds also for Proposition 8.16.

We mention here that when for no constant  $C$  the above estimate on the blow-up rate of the curvature is satisfied, the singularity is called of *Type II*.

2. As for the parabolic rescaling (see the discussion immediately after Lemma 8.18), it can be shown that the convergence of the rescaled networks  $\tilde{\mathbb{S}}_{x_0, t_j}$  to  $\tilde{\mathbb{S}}_\infty$  is locally smooth far from the cores and non multiplicity-one curves of  $\tilde{\mathbb{S}}_\infty$ .

Notice that the blow-up limit degenerate shrinker  $\tilde{\mathbb{S}}_\infty$ , obtained by this proposition, a priori depends on the chosen sequence of rescaled times  $t_j \rightarrow +\infty$ . If such a limit is a multiplicity-one line (or a halfline, if  $x_0$  is an end-point of the network), by White's local regularity theorem for mean curvature flow in [84],  $\hat{\Theta}(x_0) = 1$  ( $\hat{\Theta}(x_0) = 1/2$  in the case of a halfline) and locally around  $x_0$  the curvature is bounded, hence, the limit is unique. In general, the uniqueness of such limit is actually unknown.

**Open Problem 8.22** (Uniqueness of Blow-up Assumption – U). The limit degenerate regular shrinker  $\tilde{\mathbb{S}}_\infty$  is independent of the chosen converging sequence of rescaled networks  $\tilde{\mathbb{S}}_{x_0, t_j}$  in Proposition 8.20. More precisely, the full family  $\tilde{\mathbb{S}}_{x_0, t}$   $C_{\text{loc}}^1$ -converges to  $\tilde{\mathbb{S}}_\infty$ , as  $t \rightarrow +\infty$ .

*Remark 8.23.* An analogous (actually equivalent, in view of Remark 7.4) problem can be stated for the limit degenerate regular self-similarly shrinking flow  $\mathbb{S}_t^\infty$  given by a subsequence  $\mathbb{S}_t^{\mu_{i_j}}$  of the family of the parabolically rescaled curvature flows  $\mathbb{S}_t^{\mu_i}$  in Proposition 8.16, about the independence of  $\mathbb{S}_t^\infty$  of the chosen subsequence  $\mu_{i_j}$ , that is, the full convergence of the family of flows  $\mathbb{S}_t^\mu$  to  $\mathbb{S}_t^\infty$ .

*Remark 8.24.* The above uniqueness assumption, in case the limit degenerate regular shrinker  $\tilde{\mathbb{S}}_\infty$  is actually a multiplicity-one regular shrinker (or the same for the limit degenerate regular self-similarly shrinking flow  $\mathbb{S}_t^\infty$ ), that is, there are no cores and multiplicities, implies that the singularity is of Type I (see the above Remark 8.21). Indeed, by Lemma 8.18, the convergence of the rescaled networks to  $\tilde{\mathbb{S}}_\infty$  is smooth, which implies that the curvature is locally uniformly bounded by  $C/\sqrt{T-t}$ .

It is then natural in view of this remarks to state also the following open problems.

**Open Problem 8.25** (Non-Degeneracy of the Blow-up).

- Any blow-up limit shrinker  $\tilde{\mathbb{S}}_\infty$  different from a standard cross (see Figure 9 and Lemma 8.9), which can actually appear (see Sections 10.2 and 10.3), is non-degenerate (the same for the limit self-similarly shrinking flow  $\mathbb{S}_t^\infty$ )?
- There can be curves with multiplicity larger than one?
- If  $\tilde{\mathbb{S}}_\infty$  is degenerate, there can be any cores outside the origin?

**Open Problem 8.26** (Type I Conjecture). Every singularity is of Type I (there exists a constant  $C > 0$  such that inequality (8.11) is satisfied, for every  $t \in [0, T)$ )?

*Remark 8.27.* Even if the two rescaling procedures are different (and actually one can use the more suitable for an argument), the family of blow-up limit shrinkers  $\tilde{\mathbb{S}}_\infty$  arising from Huisken's one coincides with the family of shrinkers  $\mathbb{S}_{-1/2}^\infty$  where  $\mathbb{S}_t^\infty$  is any self-similarly shrinking curvature flow coming from Proposition 8.16. This can be easily seen, by Remark 7.4, since if  $\mathbb{S}_{-1/2}^{\mu_{i_j}} \rightarrow \mathbb{S}_{-1/2}^\infty$ , then setting

$t_i = \log(\sqrt{2}\mu_i)$  we have  $\tilde{S}_{x_0, t_i} \rightarrow \tilde{S}_{-1/2}^\infty$ , as  $i \rightarrow \infty$ , hence  $\tilde{S}_{-1/2}^\infty = \tilde{S}_\infty$  for such sequence. Viceversa, if  $\tilde{S}_{x_0, t_i} \rightarrow \tilde{S}_\infty$ , setting  $\mu_i = e^{t_i}/\sqrt{2}$ , by means of Proposition 8.16, we have a converging (not relabeled) subsequence of rescaled curvature flows  $\tilde{S}_t^{\mu_i} \rightarrow \tilde{S}_t^\infty$  such that  $\tilde{S}_{-1/2}^\infty \rightarrow \tilde{S}_\infty$ , as  $i \rightarrow \infty$ , hence  $\tilde{S}_\infty = \tilde{S}_{-1/2}^\infty$ .

Notice that in the first implication, for simplicity, we assumed the convergence at time  $t = -1/2$  of the parabolically rescaled flows, which actually is not guaranteed by Proposition 8.16. To be precise, one should argue by considering a time  $t$ , such that the sequence of networks  $\tilde{S}_t^{\mu_i}$  converges to  $\tilde{S}_t^\infty = \lambda \tilde{S}_{-1/2}^\infty$ , for some factor  $\lambda > 0$ .

#### 8.4 Blow-up limits under hypotheses on the lengths of the curves of the network

**Proposition 8.28.** *Let  $\tilde{S}_t = \bigcup_{i=1}^n \gamma^i([0, 1], t)$  be the curvature flow of a regular network with fixed end-points in a smooth, convex, bounded open set  $\Omega \subset \mathbb{R}^2$ , such that three end-points of the network are never aligned. Assume that the lengths  $L^i(t)$  of the curves of the networks satisfy*

$$\lim_{t \rightarrow T} \frac{L^i(t)}{\sqrt{T-t}} = +\infty,$$

*for every  $i \in \{1, 2, \dots, n\}$ . Then, any limit degenerate regular shrinker  $\tilde{S}_\infty$ , obtained by Proposition 8.20, if non-empty, is one of the following networks.*

*If the rescaling point belongs to  $\Omega$ :*

- *a straight line through the origin with multiplicity  $m \in \mathbb{N}$  (in this case  $\hat{\Theta}(x_0) = m$ );*
- *a standard triod centered at the origin with multiplicity 1 (in this case  $\hat{\Theta}(x_0) = 3/2$ ).*

*If the rescaling point is a fixed end-point of the evolving network (on the boundary of  $\Omega$ ):*

- *a halfline from the origin with multiplicity 1 (in this case  $\hat{\Theta}(x_0) = 1/2$ ).*

Moreover, we have

$$\lim_{j \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{\tilde{S}_{x_0, t_j}} \tilde{\rho} d\sigma = \frac{1}{\sqrt{2\pi}} \int_{\tilde{S}_\infty} \tilde{\rho} d\bar{\sigma} = \Theta_{\tilde{S}_\infty} = \hat{\Theta}(x_0), \quad (8.12)$$

and the  $L^2$ -norm of the curvature of  $\tilde{S}_{x_0, t_j}$  goes to zero in every ball  $B_R \subset \mathbb{R}^2$ , as  $j \rightarrow \infty$ .

*Proof.* We assume, by Proposition 8.20, that the sequence  $\tilde{S}_{x_0, t_j}$  of reparametrized networks converges in  $C_{\text{loc}}^1 \cap W_{\text{loc}}^{2,2}$  to the limit regular shrinker network  $\tilde{S}_\infty$  composed by  $C^\infty$  curves (with possibly multiplicity), which are actually non-degenerate as the bound from below on their lengths prevents any “collapsing” along the rescaled sequence.

If the point  $x_0 \in \mathbb{R}^2$  is distinct from all the end-points  $P^r$ , then  $\tilde{S}_\infty$  has no end-points, since they go to infinity along the rescaled sequence. If  $x_0 = P^r$  for some  $r$ , the set  $\tilde{S}_\infty$  has a single end-point at the origin of  $\mathbb{R}^2$ .

Moreover, from the lower bound on the length of the curves it follows that all the curves of  $\tilde{S}_\infty$  have infinite length, hence, by Remark 8.8, they must be pieces of straight lines from the origin, because of the uniform bound  $\mathcal{H}^1(\tilde{S}_t^{\mu_i} \cap B_R) \leq C_R$ , for every ball  $B_R \subset \mathbb{R}^2$ .

This implies that every connected component of the graph underlying  $\tilde{S}_\infty$  can contain at most one 3-point and in such case such component must be mapped to a standard triod (the 120 degrees condition must be satisfied) with multiplicity one since the sequence of converging networks are all embedded (to get in the  $C_{\text{loc}}^1$ -limit a triod with multiplicity higher than one it is necessary that the approximating networks have self-intersections). Moreover, again since the converging networks are all embedded, if a standard triod is present, a straight line or another triod cannot be there, since they would intersect transversally (see Remark 8.5). Viceversa, if a straight line is present, a triod cannot be present.

If an end-point is not present, that is, we are rescaling around a point in  $\Omega$  (not on its boundary), and a 3-point is not present, the only possibility is a straight line (possibly with multiplicity) through the origin of  $\mathbb{R}^2$ .

If an end-point is present, we are rescaling around an end-point of the evolving network, hence, by the convexity of  $\Omega$  (which contains all the networks) the limit  $\tilde{S}_\infty$  must be contained in a halfplane with

boundary a straight line  $H$  for the origin. This excludes the presence of a standard triod since it cannot be contained in any halfplane. Another halfline is obviously excluded, since they “come” only from end-points and they are all distinct. In order to exclude the presence of a straight line, we observe that the argument of Proposition 8.12 implies that, if  $\Omega_t \subset \Omega$  is the evolution by curvature of  $\partial\Omega$  keeping fixed the end-points  $P^r$ , the blow-up of  $\Omega_t$  at an end-point must be a cone spanning angle strictly less than  $\pi$  (here we use the fact that three end-points are not aligned) and  $\tilde{S}_\infty$  is contained in such a cone. It follows that  $\tilde{S}_\infty$  cannot contain a straight line.

In every case the curvature of  $\tilde{S}_\infty$  is zero everywhere and the last statement follows by the  $W_{\text{loc}}^{2,2}$ -convergence.

Finally, formula (8.12) is a special case of equation (8.9).  $\square$

*Remark 8.29.* If the two curves describing the boundary of  $\Omega$  around an end-point  $P^r$  are actually segments of the same line, that is, the three end-points are  $P^{r-1}, P^r, P^{r+1}$  aligned, the argument of Proposition 8.12 does not work and we cannot conclude that taking a blow-up at  $P^r$  we only get a halfline with unit multiplicity. It could also be possible that a straight line (possibly with multiplicity) for the origin is present, coinciding with  $H$ . Moreover, in such special case, it forces also the halfline to be contained in  $H$ , since the only way to get a line, without self-intersections in the sequence of converging networks contained in  $\Omega$ , is that the curves that are converging to the straight line “pushes” the curve getting to the end-point of the network, toward the boundary of  $\Omega$ .

With the same arguments of the proof of Proposition 8.28, an analogous proposition holds for the self-similarly shrinking limit network flow obtained by the parabolic rescaling procedure.

**Proposition 8.30.** *Under the hypotheses of Proposition 8.28, the degenerate regular self-similarly shrinking network flow  $S_t^\infty$ , obtained in Proposition 8.16 by parabolically rescaling around the point  $(x_0, T)$  in space-time, is (if non-empty) one of the following “static” flows.*

*If the rescaling point belongs to  $\Omega$ :*

- a straight line through the origin with multiplicity  $m \in \mathbb{N}$  (in this case  $\hat{\Theta}(x_0) = m$ );
- a standard triod centered at the origin with multiplicity 1 (in this case  $\hat{\Theta}(x_0) = 3/2$ ).

*If the rescaling point is a fixed end-point of the evolving network (on the boundary of  $\Omega$ ):*

- a halfline from the origin with multiplicity 1 (in this case  $\hat{\Theta}(x_0) = 1/2$ ).

**Open Problem 8.31.** Is it possible to classify in general all the possible limit degenerate shrinkers  $\tilde{S}_\infty$  or self-similarly shrinking flows  $S_t^\infty$ , obtained respectively by Huisken’s dynamical procedure or by parabolic rescaling?

*Remark 8.32.* If the evolving network is a tree, every connected component of a limit degenerate regular shrinker (possibly with multiplicities) is still a tree, hence, by Lemma 8.9 and the same argument of the proof of Proposition 8.28, such network has zero curvature and it is a union of halflines from the origin, possibly with multiplicity and a core.

*Remark 8.33.* In Section 10 we will discuss under what hypotheses, the (unscaled) evolving networks  $S_t$  converge to some limit (well-behaved) set  $S_T \subset \mathbb{R}^2$ , as  $t \rightarrow T$ , and what are the relations between such  $S_T$  and any limit degenerate shrinker  $\tilde{S}_\infty$  or self-similarly shrinking flow  $S_t^\infty$ .

## 9 Local regularity

In this section we first show that any smooth, curvature flow of regular networks is only  $C_{\text{loc}}^1$ -close to the static flow given by a standard triod, it is actually smoothly close. An important ingredient here are the estimates from Proposition 5.10, under the hypotheses (5.1), which make it possible to control the evolution of the  $L^2$ -norm of  $k$  locally.

Then, this fact with the classification of possible tangent flows from Lemma 8.10 yield a local regularity theorem which implies that locally (in space-time) around the points with Gaussian density  $3/2$ , the curvature of the evolving network is bounded.

**Lemma 9.1.** *Let  $\mathbb{S}_t^i$  for  $t \in (-1, 0)$  be a sequence of smooth curvature flows of networks with uniformly bounded length ratios (see Definition 8.14), such that  $\mathbb{S}_t^i \rightarrow \mathbb{T}$  in  $C_{\text{loc}}^1$  for almost every  $t \in (-1, 0)$ , as  $i \rightarrow \infty$ , where  $\mathbb{T}$  is the static flow given by a standard triod centered at the origin. Then, the convergence is smooth on any subset of the form  $B_R(0) \times [\tilde{t}, 0)$  where  $R > 0$  and  $-1 < \tilde{t} < 0$ .*

*Proof.* As the length ratios are uniformly bounded, the exponential decay of the backward heat kernels  $\rho_{0,0}(\cdot, t)$  and the  $C_{\text{loc}}^1$ -convergence imply that, for almost every  $-1 < t < 0$  we have

$$\int_{\mathbb{S}_t^i} \rho_{0,0}(\cdot, t) ds \rightarrow \int_{\mathbb{T}} \rho_{0,0}(\cdot, t) ds = \frac{3}{2} < +\infty,$$

hence, by (8.3) it follows that the sequence of functions

$$f_i(t) = \int_{\mathbb{S}_t^i} \left| \underline{k}_i - \frac{x^\perp}{2t} \right|^2 \rho_{0,0}(\cdot, t) ds,$$

converges to zero in  $L_{\text{loc}}^1(-1, 0)$ .

Arguing as in the proof of Proposition 8.16, we see that we can choose a further subsequence (not relabeled) such that  $\mathbb{S}_t^i \rightarrow \mathbb{T}$  in  $C_{\text{loc}}^{1,\alpha} \cap W_{\text{loc}}^{2,2}$  for all  $t \in A$  where  $A \subset (-1, 0)$  is a set of full measure. Choose  $R > 0$ ,  $\tilde{t} \in (-1, 0)$  and  $t_0 \in A$  such that  $t_0 < \tilde{t}$ . Lemma 8.18, with a compactness argument, implies that the curvature of the networks  $\mathbb{S}_t^i$  with all its derivatives are uniformly bounded and the convergence  $\mathbb{S}_t^i \rightarrow \mathbb{T}$  is smooth and uniform in  $(B_{R+1}(0) \setminus B_R(0)) \times [t_0, 0)$ . We can thus introduce three “artificial” boundary points  $P_i^r(t) \in \mathbb{S}_t^i \cap (B_{R+1}(0) \setminus B_R(0))$ ,  $r = 1, 2, 3$ , for  $t \in [t_0, 0)$  along the three rays such that the estimates (5.1) are satisfied, more precisely, we can assume that

$$\partial_s^j \lambda_i(P_i^r(t), t) = 0 \quad \text{and} \quad |\partial_s^j k_i(P_i^r(t), t)| \leq 1,$$

for all  $i \geq i_0$  and all  $j \geq 0$ .

Let  $T_1 > 0$  be the constant from Proposition 5.10 for  $M = 1$  and let  $\delta = T_1/2$ . Then, choose  $t_l \in A$ , for  $l = 1, 2, \dots, N = [\delta^{-1}] + 1$ , such that

$$t_l < t_{l+1}, \quad |t_N| \leq \delta/2 \quad \text{and} \quad |t_{l+1} - t_l| \leq \delta/2,$$

for all  $0 \leq l \leq N - 1$ .

By increasing  $i_0$ , if necessary, we can assume that

$$\int_{\mathbb{S}_{t_l}^i \cap B_{R+1}(0)} k_i^2 ds \leq 1$$

and that  $\mathbb{S}_{t_l}^i$  is  $1/100$ -close in  $C^{1,\alpha}$  to  $\mathbb{T}$  on  $B_{R+1}(0)$ , for all  $l = 0, \dots, N$  and  $i > i_0$ .

Proposition 5.12 then implies uniform estimates on  $k_i$  and all its space derivatives on  $B_R(0) \times [\tilde{t}, 0)$ , for all  $i > i_0$ . This clearly implies the convergence conclusion in the statement.  $\square$

*Remark 9.2.* With a similar argument it can be shown that if  $\mathbb{S}_t^i$  converge as above to a self-similarly shrinking regular network flow, non-degenerate and with unit multiplicity, then the convergence is smooth and uniform on any compact subset of  $\mathbb{R}^2 \times (-1, 0)$  (Lemma 8.6 in [47]).

We now show a local regularity result in the spirit of White’s result for mean curvature flow [84]. We actually follow here the alternative proof of Ecker [25, Theorem 5.6].

**Theorem 9.3** (Theorem 1.3 in [47]). *Let  $\mathbb{S}_t$  for  $t \in (T_0, T)$  be a curvature flow of a smooth, regular network in  $\mathbb{R}^2$  with uniformly bounded length ratios by some constant  $L$  (see Definition 8.14). Let  $(x_0, t_0) \in \mathbb{R}^2 \times (T_0, T)$  such that  $x_0 \in \mathbb{S}_{t_0}$ , then for every  $\varepsilon, \eta > 0$  there exists a constant  $C = C(\varepsilon, \eta, L)$  such that if*

$$\Theta_{x,t}(t - r^2) \leq \Theta_{\mathbb{S}^1} - \varepsilon, \tag{9.1}$$

*for all  $(x, t) \in B_\rho(x_0) \times (t_0 - \rho^2, t_0)$  and  $0 < r < \eta\rho$ , for some  $\rho > 0$ , where  $T_0 + (1 + \eta)\rho^2 \leq t_0 < T$ , then*

$$k^2(x, t) \leq \frac{C}{\sigma^2 \rho^2},$$

*for all  $\sigma \in (0, 1)$  and every  $(x, t)$  such that  $t \in (t_0 - (1 - \sigma)^2 \rho^2, t_0)$  and  $x \in \mathbb{S}_t \cap B_{(1-\sigma)\rho}(x_0)$ .*

*Proof.* By translation and scaling we can assume that  $x_0 = 0, t_0 = 0$  and  $\rho = 1$ . We can now follow more or less verbatim the proof of Theorem 5.6 in [25].

We argue by contradiction. Supposing that the statement is not correct we can find a sequence of smooth curvature flows of regular open networks  $\mathbb{S}_t^j$ , defined for  $t \in [-1 - \eta, 0]$ , satisfying the above conditions for every  $(x, t) \in B_1(0) \times (-1, 0)$ , but with

$$\zeta_j^2 = \sup_{\sigma \in [0, 1]} \left( \sigma^2 \sup_{t \in (-(1-\sigma)^2, 0)} \sup_{\mathbb{S}_t^j \cap B_{1-\sigma}} k_j^2 \right) \rightarrow +\infty$$

as  $j \rightarrow \infty$ .

Hence, we can find  $\sigma_j \in (0, 1]$  such that

$$\zeta_j^2 = \sigma_j^2 \sup_{t \in (-(1-\sigma_j)^2, 0)} \sup_{\mathbb{S}_t^j \cap B_{1-\sigma_j}} k_j^2$$

and  $y_j \in \mathbb{S}_{\tau_j}^j \cap \overline{B_{1-\sigma_j}}$  at a time  $\tau_j \in [-(1-\sigma_j)^2, 0]$  so that

$$\zeta_j^2 = \sigma_j^2 k_j^2(y_j, \tau_j).$$

We now take

$$\lambda_j = |k_j(y_j, \tau_j)|$$

(clearly  $\lambda_j \rightarrow +\infty$  as  $j \rightarrow \infty$ ) and define

$$\tilde{\mathbb{S}}_t^j = \lambda_j \left( \mathbb{S}_{\lambda_j^{-2}t + \tau_j}^j - y_j \right),$$

for  $t \in [-\lambda_j^2 \sigma_j^2 / 4, 0]$ , following the proof of Theorem 5.6 in [25]. We can then see that

$$0 \in \tilde{\mathbb{S}}_0^j, \quad \tilde{k}_j^2(0, 0) = 1 \tag{9.2}$$

and

$$\sup_{t \in (-\lambda_j^2 \sigma_j^2 / 4, 0)} \sup_{\tilde{\mathbb{S}}_t^j \cap B_{\lambda_j \sigma_j / 2}} \tilde{k}_j^2 \leq 4 \tag{9.3}$$

for every  $j \geq 1$ . By direct computation, we have

$$\tilde{\Theta}_{\bar{x}, \bar{t}}^j(t) = \int_{\tilde{\mathbb{S}}_t^j} \rho_{\bar{x}, \bar{t}}(\cdot, t) ds = \int_{\mathbb{S}_t^j} \rho_{y_j + \bar{x}\lambda_j^{-1}, \tau_j + \bar{t}\lambda_j^{-2}}(\cdot, t) ds = \Theta_{y_j + \bar{x}\lambda_j^{-1}, \tau_j + \bar{t}\lambda_j^{-2}}^j(t)$$

where  $t = t(t) = \tau_j + \bar{t}\lambda_j^{-2}$  and  $\Theta^j$  are the Gaussian densities relative to the flows  $\mathbb{S}_t^j$ . Since, by hypothesis,  $\Theta_{y_j + \bar{x}\lambda_j^{-1}, \tau_j + \bar{t}\lambda_j^{-2}}^j(t) \leq \Theta_{\mathbb{S}^1} - \varepsilon$  for every  $j \in \mathbb{N}$ ,  $y_j + \bar{x}\lambda_j^{-1} \in B_1(0)$  and  $\tau_j + \bar{t}\lambda_j^{-2} \in (-1, 0)$ , we conclude that  $\tilde{\Theta}_{\bar{x}, \bar{t}}^j(t) \leq \Theta_{\mathbb{S}^1} - \varepsilon$ , for  $j$  sufficiently large, for all  $(\bar{x}, \bar{t}) \in \mathbb{R}^2 \times (-\infty, 0]$  and  $-\lambda_j^2 \sigma_j^2 / 4 < t < \bar{t}$ . This implies that for every  $t \in (-\lambda_j^2 \sigma_j^2 / 4, 0)$ , we have

$$\int_{\tilde{\mathbb{S}}_t^j \cap B_R(0)} \frac{e^{R^2/4t}}{\sqrt{-4\pi t}} ds \leq \int_{\tilde{\mathbb{S}}_t^j \cap B_R(0)} \frac{e^{|x|^2/4t}}{\sqrt{-4\pi t}} ds \leq \int_{\tilde{\mathbb{S}}_t^j} \rho_{0,0}(\cdot, t) ds = \tilde{\Theta}_{0,0}^j(t) \leq \Theta_{\mathbb{S}^1} - \varepsilon,$$

hence, for  $j$  sufficiently large,

$$\mathcal{H}^1(\tilde{\mathbb{S}}_t^j \cap B_R(0)) \leq C_R(t) = e^{-R^2/4t} \sqrt{-4\pi t} (\Theta_{\mathbb{S}^1} - \varepsilon). \tag{9.4}$$

Moreover, the family of networks  $\tilde{\mathbb{S}}_t^j$  has uniformly bounded length ratios by  $L$ , since this holds for the unscaled networks and such condition is scaling invariant.

Since  $\lambda_j^2 \sigma_j^2 = \zeta_j^2 \rightarrow +\infty$ , by the length estimate (9.4), arguing as in Proposition 8.16, we see that up to a subsequence, labeled again the same, for every  $t \in (-\infty, 0)$ , we have

$$\tilde{\mathbb{S}}_t^j \rightarrow \tilde{\mathbb{S}}_t^\infty$$

in  $C_{\text{loc}}^1$  and weakly in  $W_{\text{loc}}^{2,\infty}$ , for almost every  $t \in (0, -\infty)$ , to a limit  $C^{1,1}$ -flow  $\tilde{\mathbb{S}}_t^\infty$ . Actually, the uniform bound on the curvature, everywhere in space-time, implies that such convergence holds for *every*  $t \in (-\infty, 0]$  and it is locally uniform in time. Such flow (which is not a priori a curvature flow) of networks is possibly degenerate, that is, cores and higher density lines can develop, it moves with normal velocity bounded by 4, by estimates (9.3) and it is not empty as  $0 \in \tilde{\mathbb{S}}_0^j$  for every  $j \in \mathbb{N}$ , hence  $0 \in \tilde{\mathbb{S}}_0^\infty$  also.

Because of the uniformly bounded length ratios of the family of networks  $\tilde{\mathbb{S}}_t^j$  and the exponential decay of the backward heat kernels, we can pass to the limit in the Gaussian densities, as  $j \rightarrow \infty$ , that is,

$$\tilde{\Theta}_{\bar{x}, \bar{t}}^\infty(t) = \lim_{j \rightarrow \infty} \tilde{\Theta}_{\bar{x}, \bar{t}}^j(t) = \lim_{j \rightarrow \infty} \Theta_{y_j + \bar{x}\lambda_j^{-1}, \tau_j + \bar{t}\lambda_j^{-2}}^j(t) \leq \Theta_{\mathbb{S}^1} - \varepsilon$$

for all  $(\bar{x}, \bar{t}) \in \mathbb{R}^2 \times (-\infty, 0]$  and  $t < \bar{t}$ , where we denoted with  $\tilde{\Theta}^j$  and  $\tilde{\Theta}^\infty$  the Gaussian density functions relative to the flows  $\tilde{\mathbb{S}}_t^j$  and  $\tilde{\mathbb{S}}_t^\infty$ , respectively.

Moreover,  $0 \in \tilde{\mathbb{S}}_0^j$  implies  $\hat{\Theta}^j(0, 0) \geq 1$ , hence  $\tilde{\Theta}_{0,0}^j(t) \geq \hat{\Theta}^j(0, 0) \geq 1$  for every  $t < 0$ , by monotonicity. It follows that  $\tilde{\Theta}_{0,0}^\infty(t) = \lim_{j \rightarrow \infty} \tilde{\Theta}_{0,0}^j(t) \geq 1$ , thus,

$$\hat{\Theta}^\infty(0, 0) = \lim_{t \rightarrow 0} \tilde{\Theta}_{0,0}^\infty(t) = \lim_{t \rightarrow 0} \lim_{j \rightarrow \infty} \tilde{\Theta}_{0,0}^j(t) \geq 1.$$

We want now to show that  $\tilde{\mathbb{S}}_t^\infty$  is actually a static self-similarly shrinking flow given by either a multiplicity-one line or a standard triod.

As in Section 7.1, we consider the rescaled monotonicity formula for the curvature flows  $\tilde{\mathbb{S}}_t^j$ , that is, considered  $\bar{x} \in \mathbb{R}^2$  we have

$$\tilde{\Theta}_{\bar{x},0}^j(t_1) - \tilde{\Theta}_{\bar{x},0}^j(t_2) = \int_{t_1}^{t_2} \int_{\tilde{\mathbb{S}}_s^j} \left| \tilde{k}_j - \frac{x^\perp}{2s} \right|^2 \rho_{\bar{x},0}(\cdot, s) ds ds$$

hence, passing to the limit, as  $j \rightarrow \infty$ , we get (here  $d\bar{s}$  denotes the integration with respect to the canonical measure on  $\tilde{\mathbb{S}}_t^\infty$ , counting multiplicities)

$$\tilde{\Theta}_{\bar{x},0}^\infty(t_1) - \tilde{\Theta}_{\bar{x},0}^\infty(t_2) = \lim_{j \rightarrow \infty} \int_{t_1}^{t_2} \int_{\tilde{\mathbb{S}}_s^j} \left| \tilde{k}_j - \frac{x^\perp}{2s} \right|^2 \rho_{\bar{x},0}(\cdot, s) ds ds \geq \int_{t_1}^{t_2} \int_{\tilde{\mathbb{S}}_s^\infty} \left| \tilde{k}_\infty - \frac{x^\perp}{2s} \right|^2 \rho_{\bar{x},0}(\cdot, s) d\bar{s} ds \quad (9.5)$$

for every  $t_1 < t_2 \leq 0$  and  $\bar{x} \in \mathbb{R}^2$ , by the lower semicontinuity of the  $L^2$ -integral of the curvature under the  $W_{\text{loc}}^{2,\infty}$ -weak convergence. It follows that the Gaussian density function  $\tilde{\Theta}_{\bar{x},0}^\infty(t)$  is non increasing in  $t \in (-\infty, 0]$ , then, as we know that it is uniformly bounded above by  $\Theta_{\mathbb{S}^1} - \varepsilon$ , there exists the limit

$$\hat{\Theta}_{\bar{x},0}^\infty(-\infty) = \lim_{t \rightarrow -\infty} \tilde{\Theta}_{\bar{x},0}^\infty(t) \leq \Theta_{\mathbb{S}^1} - \varepsilon.$$

Notice that  $\hat{\Theta}_{0,0}^\infty(-\infty) \geq 1$ , as we know that  $\tilde{\Theta}_{0,0}^\infty(t) \geq 1$ , for every  $t < 0$ .

Parabolically rescaling the flow  $\tilde{\mathbb{S}}_t^\infty$  around the point  $(\bar{x}, 0)$  (following the proof of Proposition 8.16) by means of inequality (9.5), the uniform bound on the curvature and the uniform bound on the length ratios, we obtain that the limit (which exists by the monotonicity of  $t \mapsto \tilde{\Theta}_{\bar{x},0}^\infty(t)$ )

$$\hat{\Theta}^\infty(\bar{x}, 0) = \lim_{t \rightarrow 0} \tilde{\Theta}_{\bar{x},0}^\infty(t) \leq \hat{\Theta}_{\bar{x},0}^\infty(-\infty) \leq \Theta_{\mathbb{S}^1} - \varepsilon$$

coincides with the Gaussian density of a limit degenerate regular shrinker (possibly empty). Being such limit bounded by  $\Theta_{\mathbb{S}^1} - \varepsilon$ , the only possibilities are 0, 1 and 3/2, by Lemma 8.10 (an empty limit, a line or a standard triod).

Since  $\tilde{\mathbb{S}}_0^\infty$  is not empty, we notice that if it contains a 3-point, let us say at  $\bar{x} \in \mathbb{R}^2$ , then by the bound on the velocity, also all the networks  $\tilde{\mathbb{S}}_t^\infty$  contain a 3-point at distance less than  $-5t$  from  $\bar{x}$ . This implies that, parabolically rescaling as above around  $\bar{x}$ , we get a limit self-similarly shrinking network flow with zero curvature and with a 3-point, then it must be a static standard triod and  $\hat{\Theta}^\infty(\bar{x}, 0) = 3/2$ . We



then take a point  $\bar{x} \in \mathbb{R}^2$  such that  $\hat{\Theta}^\infty(\bar{x}, 0)$  is maximum, hence either 1 or 3/2 by what we said above, and we consider the sequence of translated and rescaled flows for  $\tau \in (-\infty, 0]$  defined as

$$\bar{\mathbb{S}}_\tau^n = \frac{1}{\sqrt{n}} \left( \tilde{\mathbb{S}}_{n\tau}^\infty - \bar{x} \right),$$

for  $n \in \mathbb{N}$ .

This family of flows still have uniformly bounded length ratios (since this holds for the flows  $\tilde{\mathbb{S}}_t^\infty$ ) and rescaling the monotonicity formula for the flows  $\bar{\mathbb{S}}_t^n$ , for every  $\tau_1 < \tau_2 < 0$ , there holds

$$\int_{\tau_1}^{\tau_2} \int_{\bar{\mathbb{S}}_\sigma^n} \left| \bar{k}_n - \frac{x^\perp}{2\sigma} \right|^2 \rho_{0,0}(\cdot, \sigma) d\bar{s} d\sigma \leq \bar{\Theta}_{0,0}^n(\tau_1) - \bar{\Theta}_{0,0}^n(\tau_2) = \tilde{\Theta}_{\bar{x},0}^\infty(n\tau_1) - \tilde{\Theta}_{\bar{x},0}^\infty(n\tau_2) \rightarrow 0$$

as  $n \rightarrow \infty$ , since  $\lim_{t \rightarrow -\infty} \tilde{\Theta}_{\bar{x},0}^\infty(t) \rightarrow \hat{\Theta}_{\bar{x},0}^\infty(-\infty)$  as  $t \rightarrow -\infty$  (here we denoted with  $\bar{\Theta}^n$  the Gaussian density functions relative to the flows  $\bar{\mathbb{S}}_\tau^n$ ).

Then, repeating the argument of the proof of Proposition 8.16, we can extract a subsequence, not relabeled, of the flows  $\bar{\mathbb{S}}_\tau^n$  converging in  $C_{\text{loc}}^1 \cap W_{\text{loc}}^{2,2}$ , for almost every  $\tau \in (-\infty, 0)$ , to a limit self-similarly shrinking flow  $\bar{\mathbb{S}}_\tau^\infty$ , as  $n \rightarrow \infty$ , which is called “tangent flow at  $-\infty$ ” to the flow  $\tilde{\mathbb{S}}_t^\infty$ .

Since,

$$\bar{\Theta}_{0,0}^n(\tau) = \int_{\bar{\mathbb{S}}_\tau^n} \rho_{0,0}(\cdot, \tau) d\bar{s} = \int_{\tilde{\mathbb{S}}_{n\tau}^\infty} \rho_{\bar{x},0}(\cdot, n\tau) d\bar{s} = \tilde{\Theta}_{\bar{x},0}^\infty(n\tau),$$

it follows that, passing to the limit as  $n \rightarrow \infty$  (again because of the uniformly bounded length ratios and the exponential decay of the backward heat kernels), for almost every  $\tau \in (-\infty, 0)$ , there holds

$$\Theta_{\bar{\mathbb{S}}_{-1/2}^\infty} = \bar{\Theta}_{0,0}^\infty(\tau) = \lim_{n \rightarrow \infty} \tilde{\Theta}_{\bar{x},0}^\infty(n\tau) = \hat{\Theta}_{\bar{x},0}^\infty(-\infty) \leq \Theta_{\mathbb{S}^1} - \varepsilon$$

which implies that the limit flow  $\bar{\mathbb{S}}_\tau^\infty$  is not empty, as  $\hat{\Theta}_{\bar{x},0}^\infty(-\infty) \geq 1$  and it is a static self-similarly shrinking flow, given by either a multiplicity-one line or a standard triod, by Lemma 8.10.

If  $\bar{\Theta}_{0,0}^\infty(\tau) = 1$ , then  $\hat{\Theta}_{\bar{x},0}^\infty(-\infty) = 1$  which forces  $\tilde{\Theta}_{\bar{x},0}^\infty(t)$  to be constant equal to one for every  $t \in (-\infty, 0)$ , since  $\hat{\Theta}^\infty(\bar{x}, 0)$  must be equal to 1.

If  $\bar{\Theta}_{0,0}^\infty(\tau) = 3/2$ , being  $\bar{\mathbb{S}}_\tau^\infty$  a standard triod, it follows that a 3-point is present in the flow  $\tilde{\mathbb{S}}_t^\infty$ , hence also in  $\mathbb{S}_0^\infty$ . Then, if we choose  $\bar{x}$  to coincide with such 3-point, we would have  $\hat{\Theta}^\infty(\bar{x}, 0) = 3/2$  and again the Gaussian density  $\tilde{\Theta}_{\bar{x},0}^\infty(t)$  is constant equal to 3/2, for  $t \in (-\infty, 0)$ .

In both cases we conclude that  $\tilde{\mathbb{S}}_t^\infty$  is a self-similarly shrinking flow around the point  $\bar{x} \in \mathbb{R}^2$ , by formula 9.5, given by a multiplicity-one line in the first case and a standard triod in the second one.

If  $\tilde{\mathbb{S}}_t^\infty$  is a line for every  $t \in (-\infty, 0]$ , hence with zero curvature, Lemma 8.18 implies that the convergence of the flows  $\tilde{\mathbb{S}}_t^j \rightarrow \tilde{\mathbb{S}}_t^\infty$  is locally smooth. This gives a contradiction since, by formula (9.2), it would follow that  $0 \in \tilde{\mathbb{S}}_0^\infty$  and  $k_\infty^2(0, 0) = 1$ .

If  $\tilde{\mathbb{S}}_t^\infty$  is a static standard triod, then Lemma 9.1 gives a contradiction as before.  $\square$

*Remark 9.4.*

1. The result is still true if the flow is only defined on the ball  $B_{2\rho}(x_0)$  by localizing Huisken’s monotonicity formula with a suitable cut-off function. This makes the result applicable for curvature flows of networks with fixed boundary points, if one assumes that there are no boundary points in  $B_{2\rho}(x_0) \times (t_0 - (1 + \eta)\rho^2, t_0)$ . We refer the reader to Section 10 in [83] and Remark 4.16 together with Proposition 4.17 in [25].
2. By an easy contradiction argument, one can show that the bound on the curvature, together with the 120 degrees condition and assumption (9.1), imply that there is a constant  $\ell = \ell(\varepsilon, \eta, \rho) > 0$  such that for  $t \in (t_0 - (1 - \sigma)^2\rho^2, t_0)$  the length of each curve of  $\mathbb{S}_t$  which intersects  $B_{(1-\sigma)\rho}(x_0)$  is bounded from below by  $\ell \cdot \sigma\rho$ . This implies, using Theorem 5.12, corresponding scaling invariant estimates on all the higher derivatives of the curvature.



**Corollary 9.5.** *If at a point  $x_0 \in \Omega$  there holds  $\widehat{\Theta}(x_0) = 3/2$ , then the curvature is uniformly bounded along the flow  $\mathbb{S}_t$ , for  $t \in [0, T)$ , in a neighborhood of  $x_0$ .*

*Proof.* First, by Lemma 8.15, the family of networks  $\mathbb{S}_t$  has uniformly bounded length ratios. Then, as  $\widehat{\Theta}(x_0) = \widehat{\Theta}(x_0, T) = 3/2$ , by the monotonicity of  $\Theta_{x_0, T}(t)$ , there exists  $\rho_1 \in (0, 1)$  such that  $\Theta_{x_0, T}(T - \rho_1^2) < 3/2 + \delta/2$ , for some small  $\delta > 0$ . The function  $(x, t) \mapsto \Theta_{x, t}(t - \rho_1^2)$  is continuous, hence, we can find  $\rho < \rho_1$  such that if  $(x, t) \in B_\rho(x_0) \times (T - \rho^2, T)$ , then  $\Theta_{x, t}(t - \rho_1^2) < 3/2 + \delta$ , thus, by monotonicity, also  $\Theta_{x, t}(t - \rho^2) < 3/2 + \delta$ , for any  $r \in (0, \rho/2)$ , as clearly  $(t - r^2) > (t - \rho_1^2)$ .

This implies that if  $3/2 + \delta < \Theta_{\mathbb{S}^1} = \sqrt{2\pi}/e$ , for any  $t_0$  close enough to  $T$  the hypotheses of Theorem 9.3 are satisfied at  $(x_0, t_0)$ , for  $\eta = 3/4$  and  $\varepsilon = \Theta_{\mathbb{S}^1} - 3/2 - \delta > 0$ . Choosing  $\sigma = 1/2$ , we conclude that

$$k^2(x, t) \leq \frac{4C(\varepsilon, 3/4)}{\rho^2}$$

for every  $(x, t)$  such that  $t \in (t_0 - \rho^2/4, t_0)$  and  $x \in \mathbb{S}_t \cap B_{\rho/2}(x_0)$ . Since this estimate on the curvature is independent of  $t_0 < T$ , it must hold for every  $t \in (T - \rho^2/4, T)$  and  $x \in \mathbb{S}_t \cap B_{\rho/2}(x_0)$  and we are done.  $\square$

## 10 The behavior of the flow at a singular time

By means of the tools of the previous sections, we want to discuss now the behavior of the network approaching the singular time  $T$ .

We have seen in Corollary 6.10 (Theorem 6.7) that at the maximal time  $T < +\infty$  of existence of the curvature flow  $\mathbb{S}_t$  of an initial regular  $C^2$  network with fixed end-points in a smooth, convex, bounded open set  $\Omega \subset \mathbb{R}^2$ , given by Theorem 6.8, either the curvature is not bounded, as  $t \rightarrow T$ , or the inferior limit of the lengths  $L^i(t)$  of at least one curve of  $\mathbb{S}_t$  is zero, as  $t \rightarrow T$ , (see anyway Problem 6.11 about the general validity of Corollary 6.10). Hence, if all the lengths of the curves of the network are uniformly positively bounded from below, the curvature is not bounded and actually, again by Corollary 6.10, the maximum of the modulus of the curvature goes to  $+\infty$ , as  $t \rightarrow T$ . By Proposition 6.17, we also know that if the curvature is uniformly bounded, all the lengths of the curves converge as  $t \rightarrow T$ , thus at least some  $L^i(t)$  must go to zero, as  $t \rightarrow T$ .

We will then divide our analysis in the following three cases:

- all the lengths of the curves of the network are uniformly positively bounded from below and the maximum of the modulus of the curvature goes to  $+\infty$ , as  $t \rightarrow T$ ;
- the curvature is uniformly bounded along the flow and the length  $L^i(t)$  of at least one curve of  $\mathbb{S}_t$  goes to zero when  $t \rightarrow T$ ;
- the curvature is not bounded and the length of at least one curve of the network is not positively bounded from below, as  $t \rightarrow T$ .

In all the three cases, the possible blow-up limits will play a key role, with the obvious consequence that the fewer possibilities we have, the easier we can get conclusions. In particular, like when studying the evolution of a single smooth closed curve along the analogous line (see [43], for instance), it is fundamental to exclude to get blow-up limits of multiplicity larger than one, in particular “multiple lines”. For curves this can be done by means of some “embeddedness” or “non-collapsing” quantities, for instance as in [39, 43] that actually inspired the analogous one that we will discuss in Section 14. Unfortunately, for a general regular network, this is still conjectural and possibly the major open problem in the subject.

**Open Problem 10.1** (Multiplicity–One Conjecture – **M1**). Every blow-up limit shrinker arising by Huisken’s rescaling procedure or limit of parabolic rescalings at a point  $x_0 \in \overline{\Omega}$  is an embedded network with multiplicity one.

This conjecture is implied by the two equivalent statements in the following open problem.

**Open Problem 10.2** (Strong Multiplicity–One Conjecture – **SM1**/No Double–Line Conjecture – **L1**).

**SM1:** Every possible  $C_{\text{loc}}^1$ -limit of rescalings of networks of the flow is an embedded network with multiplicity one.

**L1:** A straight line with multiplicity larger than one cannot be obtained as a  $C_{\text{loc}}^1$ -limit of rescalings of networks of the flow.

While it is obvious that the first statement implies both **M1** and **L1**, the fact that the second one implies the first, can be seen as follows: if **SM1** does not hold, since the networks of the flow are all embedded, any limit of rescalings  $\mathbb{S}_i$  can lose embeddedness only if two curves in the limit network “touch” each other at some point  $x_0 \in \mathbb{R}^2$  with a common tangent (or they locally coincide, if they “produce” a piece of curve with multiplicity larger than one). Then, “slowly” dilating the networks  $\mathbb{S}_i$  around  $x_0$ , in order that the distance between such two curves and  $x_0$  still go to zero, we would get a multiplicity-two line, contradicting **L1**.

We will see in Section 14 some cases in which we are able to show that the strong multiplicity-one conjecture holds, that is,

- If during the flow the triple junctions stay uniformly far each other, then **SM1** is true.
- If the initial network has at most two triple junctions, then **SM1** is true.

*Remark 10.3.* If **M1** holds, the flow  $\mathbb{S}_t^\infty$  in Proposition 8.16 is composed of embedded, multiplicity-one networks and the same holds for the limit network  $\tilde{\mathbb{S}}_\infty$  in Proposition 8.20. In particular, under the hypotheses of Proposition 8.28, any blow-up limit network at a point  $x_0$  and singular time  $T$ , obtained by Huisken’s procedure, or self-similarly shrinking network flow, obtained by the parabolic rescaling procedure, is (if not empty) a “static” straight line through the origin (then  $\hat{\Theta}(x_0) = 1$ ) or a standard triod (then  $\hat{\Theta}(x_0) = 3/2$ ), if the rescaling point belongs to  $\Omega$ . If the rescaling point is instead a fixed end-point of the evolving network on the boundary of  $\Omega$ , then such limit can only be a single halfline from the origin (and  $\hat{\Theta}(x_0) = 1/2$ ).

Before analyzing the three situations above, we set some notation and we show some general properties of the flow at the singular time.

We let  $F : \mathbb{S} \times [0, T) \rightarrow \bar{\Omega}$ , with  $T < +\infty$ , represent the curvature flow  $\mathbb{S}_t$  of a regular network moving by curvature in its maximal time interval of smooth existence. We let  $O^1, O^2, \dots, O^m$  the 3-points of  $\mathbb{S}$ .

We define the set of *reachable points* of the flow as follows:

$$\mathcal{R} = \{x \in \mathbb{R}^2 \mid \text{there exist } p_i \in \mathbb{S} \text{ and } t_i \nearrow T \text{ such that } \lim_{i \rightarrow \infty} F(p_i, t_i) = x\}.$$

Such a set is easily seen to be closed, contained in  $\bar{\Omega}$  (hence compact as  $\Omega$  is bounded) and the following lemma holds.

**Lemma 10.4.** *A point  $x \in \mathbb{R}^2$  belongs to  $\mathcal{R}$  if and only if for every time  $t \in [0, T)$  the closed ball with center  $x$  and radius  $\sqrt{2(T-t)}$  intersects  $\mathbb{S}_t$ .*

*Proof.* One of the two implications is trivial. We have to prove that if  $x \in \mathcal{R}$ , then  $F(\mathbb{S}, t) \cap \bar{B}_{\sqrt{2(T-t)}}(x) \neq \emptyset$ . If  $x$  is one of the end-points, the result is obvious, otherwise we define the function

$$d_x(t) = \inf_{p \in \mathbb{S}} |F(p, t) - x|,$$

where, due to the compactness of  $\mathbb{S}$  the infimum is actually a minimum and as  $t \rightarrow T$ , let us say for  $t > t_x$ , it cannot be achieved at an end-point, by the assumption  $x \in \mathcal{R}$  and  $x$  different from an end-point, moreover such a maximum cannot be either achieved at a 3-point, by the 120 degrees angle condition. Since the function  $d_x : [0, T) \rightarrow \mathbb{R}$  is locally Lipschitz, we can then use Hamilton’s trick (see [37] or [61, Lemma 2.1.3]), to compute its time derivative and get (for any point  $q$ , different from an end-point, where at time  $t$  the minimum of  $|F(p, t) - x|$  is attained)

$$\begin{aligned} \partial_t d_x(t) &= \partial_t |F(q, t) - x| \geq \frac{\langle k(q, t)\nu(q, t) + \lambda(q, t)\tau(q, t), F(q, t) - x \rangle}{|F(q, t) - x|} \\ &= \frac{\langle k(q, t)\nu(q, t), F(q, t) - x \rangle}{|F(q, t) - x|} \geq -\frac{1}{d_x(t)}, \end{aligned}$$

since at a point of minimum distance the vector  $\frac{F(q,t)-x}{|F(q,t)-x|}$  is parallel to  $\nu(q,t)$ . Integrating this inequality over time, we get

$$d_x^2(t) - d_x^2(s) \leq 2(s - t) \quad \text{for } s > t > t_x.$$

We now use the hypothesis that  $x$  is reachable ( $\lim_{t_i \rightarrow T} d_x(t_i) = 0$ ) and we conclude

$$d_x^2(t) = \lim_{i \rightarrow \infty} [d_x^2(t) - d_x^2(t_i)] \leq 2 \lim_{i \rightarrow \infty} (t_i - t) = 2(T - t),$$

for every  $t > t_x$ . □

As a consequence, when we consider the blow-up limit by the Huisken's procedure of the evolving networks around points of  $\overline{\Omega}$ , we have a dichotomy among these latter. Either the limit of any sequence of rescaled networks is not empty (every rescaled network intersects the unit circle, by this lemma) and we are rescaling around a point in  $\mathcal{R}$ , or the blow-up limit is empty, since the distance of the evolving network from the point of blow-up is positively bounded below (by the very definition of  $\mathcal{R}$ ) and, rescaling, the whole network goes to infinity. Conversely, if the blow-up point belongs to  $\mathcal{R}$ , the above lemma ensures that any rescaled network contains at least one point of the closed unit ball of  $\mathbb{R}^2$ , so it cannot be empty.

We now show that, assuming the multiplicity-one conjecture, as  $t \rightarrow T$ , all the 3-points of the network  $\mathbb{S}_t$  converge.

**Lemma 10.5.** *If M1 holds, there exists a radius  $R = R(\mathbb{S}_t, x_0) > 0$ , such that if a blow-up limit regular shrinker  $\tilde{\mathbb{S}}_\infty$  (or  $\mathbb{S}_{-1/2}^\infty$ ) at the point  $x_0$  has no triple junctions in the ball  $B_R(0)$ , then it is a line for the origin of  $\mathbb{R}^2$  or the unit circle.*

*Proof.* Assume that the conclusion is false, then there is a sequence  $R_i \rightarrow +\infty$  and blow-up limit regular shrinkers  $\mathbb{S}_i$  at  $x_0$ , all different from a line or circle, such that each  $\mathbb{S}_i$  has no triple junctions in  $B_{R_i}(0)$ , for every  $i \in \mathbb{N}$ .

As we said in the discussion above, any shrinker  $\mathbb{S}_i$  must intersect the unit circle, hence, by the shrinkers equation (8.1), we can extract a subsequence of  $\mathbb{S}_i$  locally converging in  $C^1$  to a non empty limit shrinker  $\tilde{\mathbb{S}}$  without triple junctions at all. By the work of Abresch and Langer [1], then  $\tilde{\mathbb{S}}$  must be a line for the origin or the unit circle and this latter case is excluded, since, for  $i$  large enough also  $\mathbb{S}_i$  would be a circle, which is a contradiction. If the limit  $\tilde{\mathbb{S}}$  is a line, by the multiplicity-one conjecture, its multiplicity must be one, being any limit of blow-up limits of  $\mathbb{S}_t$  at the point  $x_0$  again a blow-up limit at  $x_0$ .

Then, by the second point of Lemma 8.15, the contribution of  $\mathbb{S}_i \setminus B_R(0)$  to the Gaussian density of the whole  $\mathbb{S}_i$  is small as we want, for every  $i \in \mathbb{N}$ , by choosing a value  $R$  large enough, while, for sufficiently large  $i$ , the contribution of  $\mathbb{S}_i \cap B_R(0)$  is smaller than one, as  $\mathbb{S}_i \rightarrow \tilde{\mathbb{S}}$ , which is a multiplicity-one line. Hence, we conclude that the Gaussian density of  $\mathbb{S}_i$  is close to one for sufficiently large  $i$ , then Lemma 8.10 implies that  $\mathbb{S}_i$  is also a line for the origin, which is again a contradiction and we are done. □

*Remark 10.6.* It is actually possible to find a uniform value of  $R > 0$  in this lemma, also independent of the flow  $\mathbb{S}_t$  (Tom Ilmanen, *personal communication*).

**Lemma 10.7.** *If M1 holds, there exist the limits  $x_i = \lim_{t \rightarrow T} O^i(t)$ , for  $i \in \{1, 2, \dots, m\}$  and the set  $\{x_i = \lim_{t \rightarrow T} O^i(t) \mid i = 1, 2, \dots, m\}$  coincides with the union of the set of the points  $x$  in  $\Omega$  where  $\hat{\Theta}(x) > 1$  with the set of the end-points of  $\mathbb{S}_t$  such that the curve getting there “collapses” as  $t \rightarrow T$ .*

*Proof.* Let  $\mathcal{D} = \{x \in \Omega \mid \hat{\Theta}(x) > 1\}$ ,  $\mathcal{O}(t) = \{O^1(t), O^2(t), \dots, O^m(t)\}$  and  $\mathcal{P} = \{P^1, P^2, \dots, P^l\}$ . Let  $R > 0$  be given by the previous lemma and consider a finite subset  $\overline{\mathcal{D}} \subset \mathcal{D}$ , supposing that the set

$$\mathcal{I}_{\overline{\mathcal{D}}} = \left\{ t \in [-1/2 \log T, +\infty) \mid \max_{x \in \overline{\mathcal{D}}} d(x, \mathcal{O}(t)) \geq R\sqrt{2(T - t)} \right\}$$

has infinite Lebesgue measure, there must be  $x_0 \in \overline{\mathcal{D}}$  such that

$$\mathcal{I}_{x_0} = \left\{ t \in [-1/2 \log T, +\infty) \mid d(x_0, \mathcal{O}(t)) \geq R\sqrt{2(T - t)} \right\},$$

Hence, by rescaling with Huisken's procedure around  $x_0$ , by Proposition 8.20, we can extract a sequence of times  $t_j \in \mathcal{I}_{x_0}$  such that the rescaled networks  $\tilde{\mathbb{S}}_{x_0, t_j}$  converge in the  $C_{\text{loc}}^1$  to a line for the origin of  $\mathbb{R}^2$ , since in any ball centered at the origin there cannot be 3-points, by construction of  $\mathcal{I}_{x_0}$  and we assumed **M1**. This clearly implies that  $\hat{\Theta}(x_0) = 1$ , contradicting the hypothesis  $x_0 \in \mathcal{D}$ , hence,  $\mathcal{I}_{\overline{\mathcal{D}}}$  must have finite Lebesgue measure. It is easy to see that this implies that the points of  $\overline{\mathcal{D}}$  and thus of  $\mathcal{D}$ , cannot be more than the number  $m$  of the 3-points of the evolving network  $\mathbb{S}_t$ .

If now we consider a small  $\delta > 0$ , every point  $x$  in the open set

$$\Omega_\delta = \Omega \setminus \{x \in \Omega \mid d(x, \mathcal{D} \cup \mathcal{P}) \leq \delta\}$$

satisfies  $\hat{\Theta}(x) = 1$ , hence, by compactness and White's local regularity theorem implies the curvature of the evolving network is uniformly bounded in such set. Then, as  $t \rightarrow T$ , let us say for  $t$  greater than some  $\bar{t}$ , every 3-point  $O^i(t)$ , for every  $i \in \{1, 2, \dots, m\}$ , has to "choose" a point of  $\mathcal{D} \cup \mathcal{P}$  to stay close ( $\delta$  is small and  $\mathcal{D} \cup \mathcal{P}$  is finite), otherwise it would be possible to find a subsequence of times  $t_j \rightarrow T$  such that the networks  $\mathbb{S}_{t_j}$  restricted to the set  $\Omega_\delta$ , converge (because of bounded curvature, see the proof of Proposition 10.11 for more details) to a network in  $\Omega_\delta$  with a multi-point  $x_0 \in \Omega_\delta$  and this is not possible since it would imply that  $\hat{\Theta}(x_0) \geq 3/2 > 1$  which is a contradiction with the fact that the function  $\hat{\Theta}$  is equal to one at every reachable point of  $\Omega_\delta$ .

This argument clearly implies that for every  $i \in \{1, 2, \dots, m\}$ ,  $O^i(t)$  converge to some  $x_i \in \mathcal{D} \cup \mathcal{P}$ .

Finally, if  $x \in \mathcal{D}$ , there must be a multi-point in any blow-up limit shrinker, otherwise we can only have a line that would imply  $\hat{\Theta}(x) = 1$ , against the assumption. Hence, for some  $i \in \{1, 2, \dots, m\}$  and  $t_n \rightarrow T$  there must hold  $O^i(t_n) \rightarrow x_i$  that forces  $\lim_{t \rightarrow T} O^i(t) = x_i$ .

If the curve of  $\mathbb{S}_t$  getting to an end-point  $P^r$  collapses, clearly, as before, for some  $k \in \{1, 2, \dots, m\}$  and  $t_j \rightarrow T$  there must hold  $O^k(t_j) \rightarrow P^r = x_k$  and we have the same conclusion  $\lim_{t \rightarrow T} O^k(t) = P^r = x_k$ .  $\square$

We conclude this section with a geometric construction that we will use several times in the following.

We consider the curvature flow of network  $\mathbb{S}_t$  in a strictly convex set  $\Omega$ , with fixed end-points  $\{P^1, P^2, \dots, P^l\}$  on  $\partial\Omega$ , in a maximal time interval  $[0, T)$ . We recall that as the curves composing the network are at least  $C^2$  and the boundary points are fixed, at each  $P^r$  both the velocity and the curvature are zero, namely, the compatibility conditions of order two (see definition 4.6) are satisfied.

For every end-point  $P^i$ , we define the "symmetrized" networks  $\mathbb{H}_t^i$  each one obtained as the union of  $\mathbb{S}_t$  with its reflection  $\mathbb{S}_t^{R_i}$  with respect to  $P^i$ . As the domain  $\Omega$  is strictly convex and  $\mathbb{S}_t$  is inside  $\Omega$ , this operation clearly does not introduce self-intersections in the union  $\mathbb{H}_t^i = \mathbb{S}_t \cup \mathbb{S}_t^{R_i}$  and the number of triple junctions of  $\mathbb{H}_t^i$  is exactly twice the number of  $\mathbb{S}_t$ .

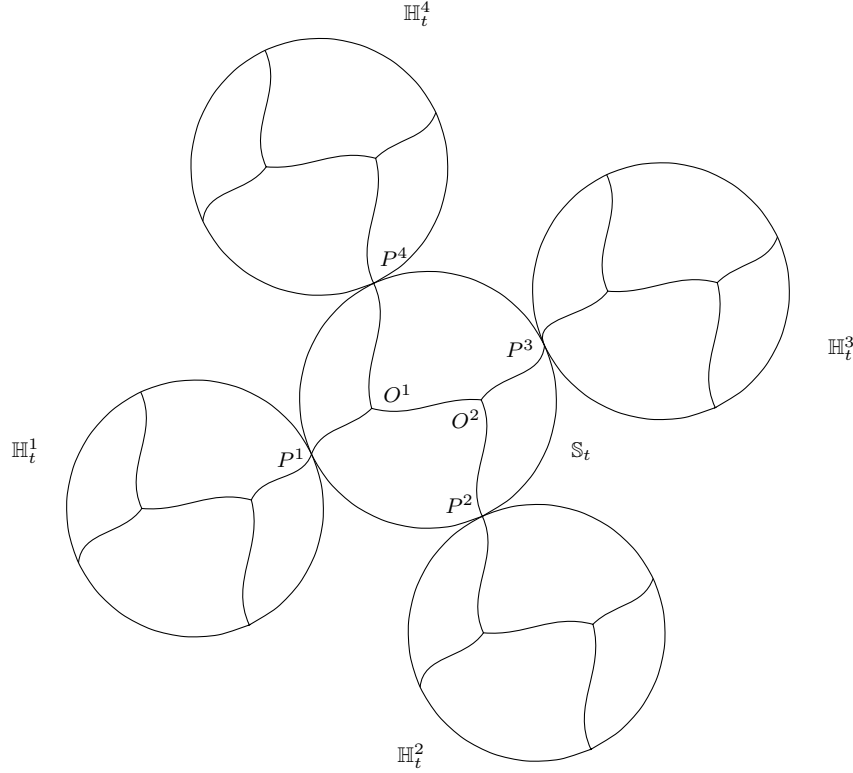


Figure 13: A network  $\mathbb{S}_t$  with the associated networks  $\mathbb{H}_t^i$ .

Every network  $\mathbb{H}_t^i$  is a regular network and the flow is still in  $C^{2,1}$ , thanks to the compatibility conditions of order two satisfied at  $P^i$ . The evolution is clearly symmetric with respect to  $P^i$ . If we have that the flow  $\mathbb{S}_t$  is smooth then also all the flows  $\mathbb{H}_t^i$  are smooth (see Definition 4.15).

### 10.1 Regularity without “vanishing” of curves

We assume that the lengths of all the curves of the network are uniformly positively bounded from below, hence the maximum of the modulus of the curvature goes to  $+\infty$ , as  $t \rightarrow T$ . We are going to show that if **M1** holds,  $T$  cannot be a singular time, hence we conclude that this case simply cannot happen. This conclusion justifies the title of this section: to have a singularity (assuming the multiplicity–one conjecture) some curve must disappear.

Such conclusion follows by the local regularity Theorem 9.3 and Remark 9.4, implying that the curvature is locally bounded around every point of  $\bar{\Omega}$ , as  $t \rightarrow T$ .

Indeed, performing a parabolic rescaling at any reachable, interior point  $x_0 \in \Omega$  (at the other interior points of  $\Omega$  the blow–up limits are empty), since we assumed that the multiplicity–one conjecture holds, by the discussion in Remark 10.3, we can obtain as blow–up limit only a straight line with unit multiplicity and  $\hat{\Theta}(x_0) = 1$  or a standard triod with  $\hat{\Theta}(x_0) = 3/2$ . By White’s local regularity theorem in [84], for the case of a straight line and the local regularity Theorem 9.3–Corollary 9.5 for the case of the standard triod (taking into account the first point of Remark 9.4 and the fact that the 3–points are converging to well separated points, by Lemma 10.7), we conclude that the curvature is uniformly locally bounded along the flow, around such point  $x_0$ .

If we instead rescale at an end–point  $P^r$  we get a halfline and this case can be treated as above by means of the “reflection construction” at the end of the previous section. That is, for the flow  $\mathbb{H}_t^r$  the point  $P^r$  is no more an end–point and a blow–up there give a straight line, hence implying that the curvature is locally bounded also around  $P^r$ , as before by White’s theorem.

By the compactness of the set of reachable points  $\mathcal{R}$ , this argument clearly implies that the curvature of  $\mathbb{S}_t$  is uniformly bounded, as  $t \rightarrow T$ , which is a contradiction.

Alternatively, performing a Huisken's rescaling at a reachable, interior point of  $\Omega$ , we obtain as blow-up limit only a straight line with unit multiplicity or a standard triod, as above. Then, instead of using the local regularity Theorem 9.3, one can argue as in [59] to show that when such limit is a regular triod, the curvature is locally bounded around such point.

**Proposition 10.8.** *Assuming M1, if  $T < +\infty$  is the maximal time interval of existence of the curvature flow of a regular network with fixed end-points, given by Theorem 6.8, then the inferior limit of the length of at least one curve is zero, as  $t \rightarrow T$ .*

*Remark 10.9.* As we conjecture (Problem 6.11) the general validity of Corollary 6.10, we expect that the conclusion of this proposition actually holds for *every* curvature flow of a regular network.

*Remark 10.10.* Proposition 10.8 can be seen as the global (in space) version of the local regularity Theorem 9.3, which deals with the situation of a single 3-point. Usually, in analytic problems local and global (in space) regularity coincide, actually in this case the tool to pass from one to the other is the validity of the multiplicity-one conjecture.

## 10.2 Limit networks with bounded curvature

The analysis in this case consists in understanding the possible limit networks that can arise, as  $t \rightarrow T$ , under the assumption that the curvature is uniformly bounded along the flow. This in order to find out how to continue the flow (if possible), which will be the argument of the next section.

As we said, at least one curve of the network  $\mathbb{S}_t$  has to “vanish”, approaching the singular time  $T$ . Anyway, we are going to show that, as  $t \rightarrow T$ , there is a unique limit degenerate regular network in  $\Omega$ , which can be *non-regular* seen as a subset of  $\mathbb{R}^2$  since a priori multi-points can appear, but anyway the sum of the exterior unit tangent vectors of the concurring curves at every multi-point must be zero, see Remark 8.2 (we recall that this implies that every “genuine” triple junction which is present still satisfies the 120 degrees condition).

**Proposition 10.11.** *If  $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i([0, 1], t)$  is the curvature flow of a regular network in  $\Omega$  with fixed end-points in a maximal time interval  $[0, T)$  such that the curvature is uniformly bounded along the flow, the networks  $\mathbb{S}_t$ , up to reparametrization proportional to arclength, converge in  $C^1$  to some degenerate regular network  $\widehat{\mathbb{S}}_T = \bigcup_{i=1}^n \widehat{\gamma}_T^i([0, 1])$  in  $\Omega$ , as  $t \rightarrow T$ .*

*Moreover, the non-degenerate curves of  $\widehat{\mathbb{S}}_T$  belong to  $C^1 \cap W^{2,\infty}$  and they are smooth outside the multi-points.*

*Proof.* As we said at the beginning of this section, by Proposition 6.17, since  $\mathbb{S}_t$  is the curvature flow of a regular network, there exist the limits of the lengths of the curves  $L^i(T) = \lim_{t \rightarrow T} L^i(t)$ , for every  $i \in \{1, 2, \dots, n\}$ . Moreover, every limit of  $\mathbb{S}_t$  is a connected, bounded subset of  $\mathbb{R}^2$ .

As the curvature and the total length are bounded by some constant  $C$ , after reparametrizing the curves  $\gamma^i$  proportional to arclength getting the maps  $\widehat{\gamma}^i$ , these latter are a family of uniformly Lipschitz maps such that  $\widehat{\gamma}_t^i$  and  $\widehat{\gamma}_{xx}^i$  are uniformly bounded in space and time by some constant  $D$ .

Then, it is easy to see that, uniformly for  $x \in [0, 1]$ , we have

$$|\widehat{\gamma}^i(x, t) - \widehat{\gamma}^i(x, \bar{t})| \leq \int_t^{\bar{t}} |\widehat{\gamma}_t^i(x, \xi)| d\xi \leq D|t - \bar{t}|,$$

which clearly means that  $\widehat{\gamma}^i(\cdot, t) : [0, 1] \rightarrow \mathbb{R}^2$  is a Cauchy sequence in  $C^0([0, 1])$ , hence the flow of reparametrized regular networks converges in  $C^1$  to a limit family of  $C^1$  curves  $\widehat{\gamma}_T^i : [0, 1] \rightarrow \mathbb{R}^2$ , as  $t \rightarrow T$ , composing the degenerate regular network  $\widehat{\mathbb{S}}_T = \bigcup_{i=1}^n \widehat{\gamma}_T^i([0, 1])$ . Clearly, by the bound on the curvature, these curves either are “constant” or belong to  $W^{2,\infty}$ , moreover, by Lemma 8.18, they are smooth outside the multi-points.

About the convergence of the unit tangent vectors, we observe that

$$\left| \frac{\partial \widehat{\gamma}^i(x, t)}{\partial x} \right| = \left| \frac{\partial \tau^i(s, t)}{\partial s} \right| L^i(t) = |k(s, t)| L^i(t) \leq C L^i(t) \leq C^2, \quad (10.1)$$

hence, every sequence of times  $t_j \rightarrow T$  have a – not relabeled – subsequence such that the maps  $\widehat{\gamma}^i(\cdot, t_j)$  converge uniformly to some maps  $\widehat{\gamma}_T^i$ .



If the curve  $\hat{\gamma}_T^i$  is a regular curve (that is,  $L^i(t)$  does not go to zero), it is easy to see that the limit maps  $\hat{\tau}_T^i$  must coincide with the unit tangent vector field  $\hat{\tau}_T^i$  to the curve  $\hat{\gamma}_T^i$ , hence, the full sequence  $\hat{\tau}^i(\cdot, t)$  converges.

If  $L^i(t)$  converges to zero, as  $t \rightarrow T$ , by inequality (10.1), the maps  $\hat{\tau}^i(\cdot, t_j)$  converge to a constant unit vector  $\hat{\tau}_T^i$  which, if independent of the subsequence  $t_j$ , will be the “assigned” constant unit vector to the degenerate constant curve  $\hat{\gamma}_T^i$  of the degenerate regular network  $\mathbb{S}$ , as in Definition 8.1.

We claim that  $\hat{\mathbb{S}}_T$  contains at least one regular non-degenerate curve, otherwise, as  $t \rightarrow T$ , the whole network  $\mathbb{S}_t$  is contracting at a single point, this clearly can happen only if the network has no end-points and the radius of the smallest ball containing it is going to zero as  $t \rightarrow T$ . Being this ball tangent to the network  $\mathbb{S}_t$  at some interior point of a curve, at such point the curvature of the network must be larger or equal to the inverse of the radius of such ball and this is a contradiction, by the uniform bound on the curvature.

If now we consider the set of the regular non-degenerate curves of  $\hat{\mathbb{S}}_T$ , their end-points contain all the constant images of the degenerate curves and the Herring condition determines the “unit tangent vectors” of the (one or two) degenerate curves concurring there (the mutual position, if these are two, is uniquely determined by the embeddedness of the converging regular networks). This argument can be iterated, considering now the degenerate concurring curves with respect to the previous degenerate curves and so on, to determine uniquely the unit tangent vectors at all the 3-points of the limit degenerate regular network  $\hat{\mathbb{S}}_T$ . Hence, the limit degenerate unit tangent vectors of  $\hat{\gamma}_T^i$  are independent of the chosen sequence of times  $t_j \rightarrow T$  and we are done.  $\square$

*Remark 10.12.* In the special situation that no curves collapse, as  $t \rightarrow T$  (we actually conjecture that this cannot happen, that is, that Problem 6.11 has a positive answer and Corollary 6.10 applies), the limit network  $\hat{\mathbb{S}}_T$  is a regular network in  $W^{2,\infty}$ , hence, one can use the extension of Theorem 6.8 mentioned at point 5 of Remark 6.9, in order to continue the flow after the time  $T$ . In this very “strange” case, one should investigate if this “extended” curvature flow, which is  $C^{2,1}$  with the exception of time  $t = T$ , is actually always  $C^{2,1}$ , getting a contradiction by the maximality of the interval of  $C^{2,1}$  existence  $[0, T]$ .

If we consider the family of the non-degenerate curves of  $\hat{\mathbb{S}}_T$ , they describe a  $C^1$  network, that we call  $\mathbb{S}_T$ , which is not necessarily a regular network (it can have multi-points), but by Remark 8.2, the sum of the exterior unit tangent vectors of the concurring curves at every multi-point in  $\Omega$  must still be zero.

*Remark 10.13.* Notice that, even if  $\mathbb{S}_T$  is smooth outside its multi-points and  $W^{2,\infty}$ , we cannot say at the moment that its curves are of class  $C^2$ . This will be actually a consequence of the analysis of the next section, see Remark 10.27.

We want to show now that assuming the multiplicity-one conjecture, the only possible “collapsing” situation we have to deal with in the interior of  $\Omega$  is given by two 3-points converging each other along a single “collapsing” curve “producing” a 4-point with two pairs of curves with opposite exterior unit tangents forming four angles of 120, 60, 120 and 60 degrees (in this case the core of the limit degenerate regular network is given only by the “collapsed” curve). Analogously, at any end-point there can only be a “collapse” of the curve of the network getting there, getting in the limit two curves “exiting” from such end-point and forming an angle of 120 degrees among them (again the core is still given only by the collapsed curve).

**Proposition 10.14.** *If M1 is true, every multi-point of the network  $\mathbb{S}_T$  is either a regular triple junction or an end-point of  $\mathbb{S}_t$  or*

- *a 4-point where the four concurring curves have opposite exterior unit tangent vectors in pairs and form angles of 120/60 degrees between them – collapse of a curve in the “interior” of  $\mathbb{S}_t$ ,*
- *a 2-point at an end-point of the network  $\mathbb{S}_t$  where the two concurring curves form an angle of 120 degrees among them – collapse of the curve getting to such end-point of  $\mathbb{S}_t$ .*



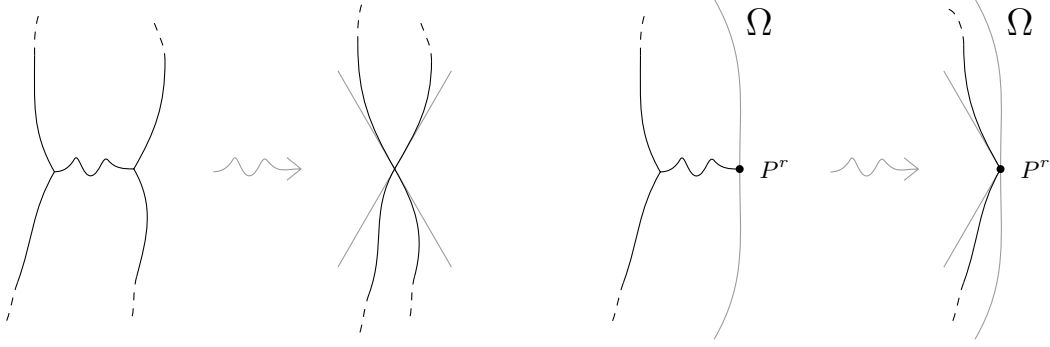


Figure 14: Collapse of a curve in the interior and at an end-point of  $\mathbb{S}_t$ .

*Proof.* Since the curvature is bounded, no regions can collapse, by the computations in Section 8.2, hence, around every point the network is a tree, as  $t$  gets close to  $T$ . Assuming that the vertex belongs to  $\Omega$ , we can follow the argument in the proof of the second part of Lemma 8.9, to show that the core of  $\widehat{\mathbb{S}}_T$  must be a single curve and we have in  $\mathbb{S}_T$  a 4-point where the four concurring curves have opposite unit tangents in pairs and form angles of  $120/60$  degrees between them. The only extra fact we have to show is that  $\mathbb{S}_T$  cannot have a multi-point  $O_T$  with two concurring curves with the same exterior unit tangent vector. But in such case, by rescaling  $\mathbb{S}_t$  around  $O_T$ , we would get a blow-up limit network composed only by halflines (the curvature of  $\mathbb{S}_t$  is bounded, hence any blow-up limit must have zero curvature) with one of them with multiplicity two (at least), contradicting the multiplicity-one conjecture **M1**, that we assumed to hold.

In the case a vertex of  $\mathbb{S}_T$  coincides with an end-point  $P^r$  of  $\mathbb{S}_t$ , we get the statement, by considering the network  $\mathbb{H}_t^r$ , obtained by the union of  $\mathbb{S}_t$  with its reflection with respect to the point  $P^r$  (see the discussion just before Section 10.1) and applying the previous conclusion to such network.  $\square$

*Remark 10.15.* It follows that every core (there could be more than one) of  $\widehat{\mathbb{S}}_T$  is composed by a single “collapsed” curve.

*Remark 10.16.* Notice that if at an end-point the two curves of the boundary of the convex set  $\Omega$  form an angle (or the whole network is contained in an angle whose vertex is such end-point) with amplitude less than  $120$  degrees, then the collapse situation described in Proposition 10.14 cannot happen at such end-point. This is, for instance, the case of an initial triod contained in a triangle with angles less than  $120$  degrees and fixed end-points in the vertices.

The same conclusion holds, by the argument in the proof of Proposition 8.12, calling  $\Omega_t \subset \Omega$  the evolution by curvature of  $\partial\Omega$ , keeping fixed the end-points of  $\mathbb{S}_t$ , if the angle formed by  $\Omega_t$  at such end-point, becomes smaller than  $120$  degrees.

**Corollary 10.17.** *If **M1** holds and the curvature of  $\mathbb{S}_t$  is uniformly bounded during the flow, the networks  $\mathbb{S}_t$ , up to reparametrization, converge in  $C^1$  to some degenerate regular network  $\widehat{\mathbb{S}}_T$ , whose non-degenerate curves form a  $C^1$  network  $\mathbb{S}_T$ , having all its multi-points which are among the ones described in Proposition 10.14. Moreover, the curves of  $\mathbb{S}_T$  belong to  $C^1 \cap W^{2,\infty}$  and are smooth outside the multi-points.*

All the previous arguments can be easily localized and we have the following conclusion.

**Proposition 10.18.** *If **M1** holds and the curvature of  $\mathbb{S}_t$  is locally uniformly bounded around a point  $x_0 \in \overline{\Omega}$ , as  $t \rightarrow T$ , the networks  $\mathbb{S}_t$ , up to reparametrization, converge in  $C_{\text{loc}}^1$  locally around  $x_0$  to some degenerate regular network  $\widehat{\mathbb{S}}_T$  whose non-degenerate curves form a  $C^1$  network  $\mathbb{S}_T$ , having a possibly non-regular multi-point at  $x_0$  which is among the ones described in Proposition 10.14. Moreover, the curves of  $\mathbb{S}_T$  belong to  $C^1 \cap W^{2,\infty}$ , in a neighborhood of  $x_0$ , and are smooth outside the multi-point.*

*Remark 10.19.* Referring to Remark 8.21, we can call these singularities with bounded curvature *Type 0* singularities. They are peculiar of the network flow, as they cannot appear in the motion by curvature of a single curve.

### 10.3 Vanishing of curves with unbounded curvature

The last case, when, as  $t \rightarrow T$ , the curvature is not bounded and the length of at least one curve of the curvature flow  $\mathbb{S}_t$  of a regular network with fixed end-points, given by Theorem 6.8, is not positively bounded from below, is the most delicate. Performing, as before, any of the blow-up procedures, even assuming the multiplicity-one conjecture, there can be several shrinkers as possible blow-up limits given by Propositions 8.16, 8.20 and we need to classify them in order to understand the behavior of the flow  $\mathbb{S}_t$  approaching the singular time  $T$ . In doing that the (local) structure (topology) of the evolving network plays an important role in the analysis, since it restricts the family of possible shrinkers obtained as blow-up limits of  $\mathbb{S}_t$ .

A very relevant case is when the evolving network is a tree, that is, it has no loops.

**Proposition 10.20.** *If **M1** holds and the evolving regular network  $\mathbb{S}_t$  is a tree in a neighborhood of  $x_0 \in \overline{\Omega}$ , for  $t$  close enough to  $T$ , then the curvature of  $\mathbb{S}_t$  is locally uniformly bounded around  $x_0$ , during the flow.*

*Proof.* Let  $\mathbb{S}_t$  be a smooth flow in the maximal time interval  $[0, T)$  of the initial network  $\mathbb{S}_0$ . Let  $x_0 \in \overline{\Omega}$  be a reachable point for the flow and let  $B$  be a ball containing  $x_0$  where  $\mathbb{S}_t$  is a tree, for  $t$  close enough to  $T$  (we clearly only need to consider reachable points).

Let us consider a sequence of parabolically rescaled curvature flows  $\mathbb{S}_t^{\mu_i}$  around  $(x_0, T)$ , as in Proposition 8.16. Then, as  $i \rightarrow \infty$ , it converges to a degenerate regular self-similarly shrinking network flow  $\mathbb{S}_t^\infty$ , in  $C_{\text{loc}}^{1,\alpha} \cap W_{\text{loc}}^{2,2}$ , for almost all  $t \in (-\infty, 0)$  and for any  $\alpha \in (0, 1/2)$ .

Thanks to the multiplicity-one hypothesis **M1** and to the topology of the network (locally a tree, see Lemma 8.9), if we suppose that  $x_0 \notin \partial\Omega$ , then  $\mathbb{S}_t^\infty$  can only be the “static” flow given by:

- a straight line;
- a standard triod;
- four concurring halflines with opposite unit tangent vectors in pairs, forming angles of  $120/60$  degrees between them, that is, a standard cross.

By White’s local regularity theorem in [84], if the sequence of rescaled curvature flows converges to a straight line, the curvature is uniformly bounded for  $t \in [0, T)$  in a ball around the point  $x_0$ . Thanks to Theorem 9.3 the same holds in the case of the standard triod. Hence, the only situation we have to deal with to complete the proof in this case is the collapse of two triple junctions at a point of  $\Omega$ , when the limit flow is given by the static degenerate regular network composed by four concurring halflines with opposite unit tangents in pairs forming angles of  $120/60$  degrees between them, a standard cross. We claim that also in this case the curvature is locally uniformly bounded during the flow, around the point  $x_0$  (the next proposition and lemmas are devoted to prove this fact).

If instead  $x_0 \in \partial\Omega$ , the only two possibilities for  $\mathbb{S}_t^\infty$  are the static flows given by:

- a halfline;
- two concurring halflines forming an angle of  $120$  degrees.

For both these two situation the thesis is obtained by going back to the case in which  $x_0 \in \Omega$ , with the “reflection construction” we described just before Section 10.1.  $\square$

**Proposition 10.21.** *Let  $\mathbb{S}_t$  be a smooth flow in the maximal time interval  $[0, T)$  for the initial network  $\mathbb{S}_0$ . Let  $x_0$  be a reachable point for the flow such that the sequence of rescaled curvature flows  $\mathbb{S}_t^{\mu_i}$  around  $(x_0, T)$ , as in Proposition 8.16, as  $i \rightarrow \infty$ , converges, in  $C_{\text{loc}}^{1,\alpha} \cap W_{\text{loc}}^{2,2}$ , for almost all  $t \in (-\infty, 0)$  and for any  $\alpha \in (0, 1/2)$ , to a limit degenerate static flow  $\mathbb{S}_t^\infty$  given by a standard cross. Then,*

$$|k(x, t)| \leq C < +\infty$$

for all  $t \in [0, T)$  and  $x$  in a neighborhood of  $x_0$ .

We briefly outline the proof of this proposition. First, in Lemma 10.22 and 10.23, we show that for any tree, if we assume a uniform control on the motion of its end-points, the  $L^2$ -norm of its curvature is uniformly bounded in a time interval depending on its initial value. Moreover, we also bound the

$L^\infty$ -norm of the curvature in terms of its  $L^2$ -norm and of the  $L^2$ -norm of its derivative.

Then, we prove that for a special tree, composed by only five curves, two triple junctions and four fixed end-points on the boundary of  $\Omega$  open, convex and regular (see Figure 15), uniformly controlling, as before, its end-points and the lengths of the “boundary curve” from below, the  $L^2$ -norm of  $k_s$  is bounded until  $\|k\|_{L^2}$  stays bounded. The statement of the proposition will follow by localizing these estimates.

**Lemma 10.22.** *Let  $\Omega$  be a convex open regular set and  $\mathbb{S}_0$  a tree with end-points  $P^1, P^2, \dots, P^l$  (not necessarily fixed during its motion) on  $\partial\Omega$ . Let  $\mathbb{S}_t$  be a smooth evolution by curvature for  $t \in [0, T)$  of the network  $\mathbb{S}_0$  such that the square of the curvature at the end-points of  $\mathbb{S}_t$  is uniformly bounded in time by some constant  $C$ . Then,*

$$\|k\|_{L^\infty}^2 \leq 4^{n-1}C + D_n\|k\|_{L^2}\|k_s\|_{L^2}, \quad (10.2)$$

where  $n \in \mathbb{N}$  is such that for every point  $Q \in \mathbb{S}_0$  there is a path to get from  $Q$  to an end-point passing by at most  $n$  curves (clearly,  $n$  is smaller than the total number of curves of  $\mathbb{S}_0$ ) and the constant  $D_n$  depends only on  $n$ .

*Proof.* Let us first consider a network  $\mathbb{S}_0$  with five curves, two triple junctions  $O^1, O^2$  and four end-points  $P^1, P^2, P^3, P^4$ . In this case  $n$  is clearly equal to two. We call  $\gamma^i$ , for  $i \leq 4$ , the curve connecting  $P^i$  with one of the two triple junctions and  $\gamma^5$  the curve connecting the two triple junctions (see the following Figure 15).

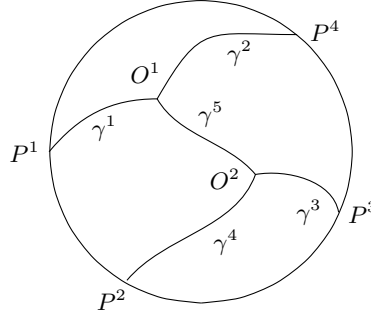


Figure 15: A tree-like network with five curves.

Fixed a time  $t \in [0, T)$ , let  $Q \in \gamma^i \subset \mathbb{S}_t$ , for some  $i \leq 4$ . We compute

$$[k^i(Q)]^2 = [k^i(P^i)]^2 + 2 \int_{P^i}^Q k k_s ds \leq C + 2\|k\|_{L^2}\|k_s\|_{L^2},$$

hence, for every  $Q \in \mathbb{S}_t \setminus \gamma^5$  we have

$$[k^i(Q)]^2 \leq C + 2\|k\|_{L^2}\|k_s\|_{L^2}.$$

Assume now instead that  $Q \in \gamma^5$ . Recalling that  $\sum_{i=1}^3 k^i = 0$  at each triple junction, by the previous argument we have  $[k^i(O^1)]^2, [k^i(O^2)]^2 \leq C + 2\|k\|_{L^2}\|k_s\|_{L^2}$ , for all  $i \in \{1, 2, 3, 4\}$ , then it follows that  $[k^5(O^1)]^2, [k^5(O^2)]^2 \leq 4C + 8\|k\|_{L^2}\|k_s\|_{L^2}$ . Hence, arguing as before, we get

$$[k^5(Q)]^2 = [k^5(O^1)]^2 + 2 \int_{O^1}^Q k k_s ds \leq 4C + 8\|k\|_{L^2}\|k_s\|_{L^2} + 2 \int_{O^1}^Q k k_s ds,$$

In conclusion, we get the uniform in time inequality for  $\mathbb{S}_t$

$$\|k\|_{L^\infty}^2 \leq 4C + 10\|k\|_{L^2}\|k_s\|_{L^2}.$$

In the general case, since  $\mathbb{S}_t$  are all trees homeomorphic to  $\mathbb{S}_0$ , we can argue similarly to get the conclusion by induction on  $n$ .  $\square$

**Lemma 10.23.** Let  $\Omega \subset \mathbb{R}^2$  be open, convex and regular, let  $\mathbb{S}_0$  be a tree with end-points  $P^1, P^2, \dots, P^l$  on  $\partial\Omega$  that satisfy assumptions (5.1) and let  $\mathbb{S}_t$  for  $t \in [0, T)$  be a smooth evolution by curvature of the network  $\mathbb{S}_0$ . Then  $\|k\|_{L^2}^2$  is uniformly bounded on  $[0, \tilde{T}]$  by  $\sqrt{2}[\|k(\cdot, 0)\|_{L^2}^2 + 1]$ , where

$$\tilde{T} = \min \left\{ T, 1/8C (\|k(\cdot, 0)\|_{L^2}^2 + 1)^2 \right\}.$$

Here the constant  $C$  depends only on the number  $n \in \mathbb{N}$  of Lemma 10.22 and the constants in assumptions (5.1).

*Proof.* By inequality (5.4) we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{S}_t} k^2 ds &\leq -2 \int_{\mathbb{S}_t} k_s^2 ds + \int_{\mathbb{S}_t} k^4 ds + \sum_{p=1}^m \sum_{i=1}^3 \lambda^{pi} (k^{pi})^2 \Big|_{\text{at the 3-point } O^p} + C \\ &\leq -2 \int_{\mathbb{S}_t} k_s^2 ds + \|k\|_{L^\infty}^2 \int_{\mathbb{S}_t} k^2 ds + C\|k\|_{L^\infty}^3 + C. \end{aligned} \quad (10.3)$$

By estimate (10.2) and the Young inequality, we then obtain

$$\begin{aligned} \|k\|_{L^\infty}^3 &\leq C_n + C_n \|k\|_{L^2}^{\frac{3}{2}} \|k_s\|_{L^2}^{\frac{3}{2}} \leq C_n + \varepsilon \|k_s\|_{L^2}^2 + C_{n,\varepsilon} \|k\|_{L^2}^6, \\ \|k\|_{L^\infty}^2 \|k\|_{L^2}^2 &\leq C_n \|k\|_{L^2}^2 + D_n \|k\|_{L^2}^3 \|k_s\|_{L^2} \leq C_n \|k\|_{L^2}^2 + \varepsilon \|k_s\|_{L^2}^2 + C_{n,\varepsilon} \|k\|_{L^2}^6, \end{aligned}$$

for every small  $\varepsilon > 0$  and a suitable constant  $C_{n,\varepsilon}$ .

Plugging these estimates into inequality (10.3) we get

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{S}_t} k^2 ds &\leq -2 \|k_s\|^2 + \|k\|_{L^\infty}^2 \|k\|^2 + C\|k\|_{L^\infty}^3 + C \\ &\leq -2 \|k_s\|^2 + C_n \|k\|_{L^2}^2 + \varepsilon \|k_s\|_{L^2}^2 + C_{n,\varepsilon} \|k\|_{L^2}^6 + C_n + \varepsilon \|k_s\|_{L^2}^2 + C_{n,\varepsilon} \|k\|_{L^2}^6 + C_n \\ &\leq C \left( \int_{\mathbb{S}_t} k^2 ds \right)^3 + C, \end{aligned} \quad (10.4)$$

Where we chose  $\varepsilon = 1/2$  and the constant  $C$  depends only on the number  $n \in \mathbb{N}$  of Lemma 10.22 and the constants in conditions (5.1).

Calling  $y(t) = \int_{\mathbb{S}_t} k^2 ds + 1$ , we can rewrite inequality (10.4) as the differential ODE

$$y'(t) \leq 2C y^3(t),$$

hence, after integration, we get

$$y(t) \leq \frac{1}{\sqrt{\frac{1}{y^2(0)} - 4Ct}}$$

and, choosing  $\tilde{T}$  as in the statement, the conclusion is straightforward.  $\square$

**Lemma 10.24.** Let  $\Omega \subset \mathbb{R}^2$  be open, convex and regular, let  $\mathbb{S}_0$  be a tree with five curves, two triple junctions  $O^1, O^2$  and four end-points  $P^1, P^2, P^3, P^4$  on  $\partial\Omega$ , as in Figure 15, satisfying assumptions (5.1) and assume that  $\mathbb{S}_t$ , for  $t \in [0, T)$ , is a smooth evolution by curvature of the network  $\mathbb{S}_0$  such that  $\|k\|_{L^2}$  is uniformly bounded on  $[0, T)$ .

If the lengths of the curves of the network arriving at the end-points are uniformly bounded below by some constant  $L > 0$ , then  $\|k_s\|_{L^2}$  is uniformly bounded on  $[0, T)$ .

*Proof.* We first estimate  $\|k_s\|_{L^\infty}^2$  in terms of  $\|k_s\|_{L^2}$  and  $\|k_{ss}\|_{L^2}$ .

Fixed a time  $t \in [0, T)$ , let  $Q \in \gamma^i \subset \mathbb{S}_t$ , for some  $i \leq 4$ . We compute

$$[k_s^i(Q)]^2 = [k_s^i(P^i)]^2 + 2 \int_{P^i}^Q k_s k_{ss} ds \leq C + 2 \|k_s\|_{L^2} \|k_{ss}\|_{L^2},$$

hence, in this case,

$$[k_s^i(Q)]^2 \leq C + 2 \|k_s\|_{L^2} \|k_{ss}\|_{L^2},$$

for every  $Q \in \mathbb{S}_t \setminus \gamma^5$ .

Assume now instead that  $Q \in \gamma^5$ . Recalling that  $k_s^i + \lambda^i k^i = k_s^j + \lambda^j k^j$  at each triple junction, we get

$$k_s^5(O^1) = k_s^i(O^1) + \lambda^i(O^1)k^i(O^1) - \lambda^5(O^1)k^5(O^1),$$

hence,

$$\begin{aligned} |k_s^5(O^1)| &\leq |k_s^i(O^1)| + C\|k\|_{L^\infty}^2 \\ &\leq |k_s^i(O^1)| + C\|k\|_{L^2}\|k_s\|_{L^2} + C \\ &\leq |k_s^i(O^1)| + C(1 + \|k_s\|_{L^2}), \end{aligned}$$

by Lemma 10.23. Then,

$$[k_s^5(O^1)]^2 \leq 2[k_s^i(O^1)]^2 + C(1 + \|k_s\|_{L^2}^2)$$

and it follows

$$\begin{aligned} [k_s^5(Q)]^2 &= [k_s^5(O^1)]^2 + 2 \int_{O^1}^Q k_s k_{ss} ds \\ &\leq 2[k_s^i(O^1)]^2 + C(1 + \|k_s\|_{L^2}^2) + 2 \int_{O^1}^Q k_s k_{ss} ds \\ &\leq C + C\|k_s\|_{L^2}^2 + 2\|k_s\|_{L^2}\|k_{ss}\|_{L^2}, \end{aligned}$$

since, by the previous argument, we have  $[k_s^i(O^1)]^2, [k_s^i(O^2)]^2 \leq C + 2\|k_s\|_{L^2}\|k_{ss}\|_{L^2}$ , for all  $i \in \{1, 2, 3, 4\}$ . Hence, we conclude

$$\|k_s\|_{L^\infty}^2 \leq C + C\|k_s\|_{L^2}^2 + 2\|k_s\|_{L^2}\|k_{ss}\|_{L^2}.$$

We now pass to estimate  $\|k_s\|_{L^2}$ . Making computation (5.3) explicit for  $j = 1$ , we have

$$\partial_t \int_{\mathbb{S}_t} k_s^2 ds \leq -2 \int_{\mathbb{S}_t} k_{ss}^2 ds + 7 \int_{\mathbb{S}_t} k^2 k_{ss}^2 ds - \sum_{p=1}^2 \sum_{i=1}^3 2k_s^{pi} k_{ss}^{pi} + \lambda^{pi} (k_s^{pi})^2 \Big|_{\text{at the 3-point } O^p} + C. \quad (10.5)$$

Then, as in Section 5 we work to lower the differentiation order of the boundary term  $\sum_{i=1}^3 k_s^i k_{ss}^i$  at each 3-point.

We claim that the following equality holds at each 3-point,

$$3 \sum_{i=1}^3 \lambda^i k^i k_t^i = \partial_t \sum_{i=1}^3 \lambda^i (k^i)^2. \quad (10.6)$$

Keeping in mind that, at every 3-point, we have  $\sum_{i=1}^3 k^i = 0$  and  $\lambda^i = \frac{k^{i-1} - k^{i+1}}{\sqrt{3}}$ , with the convention that the superscripts are considered “modulus 3” (see Section 3), we obtains

$$\begin{aligned} \sqrt{3} \sum_{i=1}^3 \lambda^i k^i k_t^i &= \sum_{i=1}^3 (k^{i-1} - k^{i+1}) k^i k_t^i \\ &= \sum_{i=1}^3 k^{i+1} (k^{i+1} + k^{i-1}) k_t^i - k^{i-1} (k^{i+1} + k^{i-1}) k_t^i \\ &= \sum_{i=1}^3 \left[ (k^{i+1})^2 - (k^{i-1})^2 \right] k_t^i, \end{aligned}$$

and

$$\begin{aligned}
\sqrt{3}\partial_t \sum_{i=1}^3 \lambda^i (k^i)^2 &= \sqrt{3} \sum_{i=1}^3 \lambda_t^i (k^i)^2 + 2\lambda^i k^i k_t^i \\
&= \sum_{i=1}^3 (k_t^{i-1} - k_t^{i+1}) (k^i)^2 + 2 \sum_{i=1}^3 (k^{i-1} - k^{i+1}) k^i k_t^i \\
&= \sum_{i=1}^3 \left[ (k^{i+1})^2 - (k^{i-1})^2 + 2k^i k^{i-1} - 2k^i k^{i+1} \right] k_t^i \\
&= \sum_{i=1}^3 \left[ (k^{i+1})^2 - (k^{i-1})^2 - 2(k^{i-1} + k^{i+1})k^{i-1} + 2(k^{i-1} + k^{i+1})k^{i+1} \right] k_t^i \\
&= 3 \sum_{i=1}^3 \left[ (k^{i+1})^2 - (k^{i-1})^2 \right] k_t^i,
\end{aligned}$$

thus, equality (10.6) is proved.

Now we use such equality to lower the differentiation order of the term  $\sum_{i=1}^3 k_s^i k_{ss}^i$ . Recalling the formula  $\partial_t k = k_{ss} + k_s \lambda + k^3$  and that  $\sum_{i=1}^3 k_t^i = \partial_t \sum_{i=1}^3 k^i = 0$ , we get

$$\begin{aligned}
\sum_{i=1}^3 k_s^i k_{ss}^i &= \sum_{i=1}^3 k_s^i [k_t^i - \lambda^i k_s^i - (k^i)^3] \\
&= \sum_{i=1}^3 (k_s^i + \lambda^i k^i - \lambda^i k^i) k_t^i - \sum_{i=1}^3 \lambda^i (k_s^i)^2 + (k^i)^3 k_s^i \\
&= \sum_{i=1}^3 (k_s^i + \lambda^i k^i) k_t^i - \sum_{i=1}^3 \lambda^i k^i k_t^i - \sum_{i=1}^3 \lambda^i (k_s^i)^2 + (k^i)^3 k_s^i \\
&= -\partial_t \sum_{i=1}^3 \lambda^i (k^i)^2 / 3 - \sum_{i=1}^3 \lambda^i (k_s^i)^2 + (k^i)^3 k_s^i,
\end{aligned}$$

at the triple junctions  $O^1$  and  $O^2$ , where we used the fact that  $k_s^i + \lambda^i k^i$  is independent of  $i \in \{1, 2, 3\}$ . Substituting this equality into estimate (10.5), we obtain

$$\begin{aligned}
\partial_t \int_{\mathbb{S}_t} k_s^2 ds &\leq -2 \int_{\mathbb{S}_t} k_{ss}^2 ds + 7 \int_{\mathbb{S}_t} k^2 k_s^2 ds + \sum_{p=1}^2 \sum_{i=1}^3 2 (k^{pi})^3 k_s^{pi} + \lambda^{pi} (k_s^{pi})^2 \Big|_{\text{at the 3-point } O^p} + C \\
&\quad + 2\partial_t \sum_{p=1}^2 \sum_{i=1}^3 \lambda^{pi} (k^{pi})^2 / 3 \Big|_{\text{at the 3-point } O^p} \\
&\leq -2 \int_{\mathbb{S}_t} k_{ss}^2 ds + C \|k\|_{L^2}^2 \|k_s\|_{L^\infty}^2 + \sum_{p=1}^2 \sum_{i=1}^3 2 (k^{pi})^3 k_s^{pi} + \lambda^{pi} (k_s^{pi})^2 \Big|_{\text{at the 3-point } O^p} \\
&\quad + 2\partial_t \sum_{p=1}^2 \sum_{i=1}^3 \lambda^{pi} (k^{pi})^2 / 3 \Big|_{\text{at the 3-point } O^p} + C. \tag{10.8}
\end{aligned}$$

Using the previous estimate on  $\|k_s\|_{L^\infty}$ , the hypothesis of uniform boundedness of  $\|k\|_{L^2}$  and Young inequality, we get

$$\begin{aligned}
\|k\|_{L^2}^2 \|k_s\|_{L^\infty}^2 &\leq C + C \|k_s\|_{L^2}^2 + C \|k_s\|_{L^2} \|k_{ss}\|_{L^2} \\
&\leq C + C \|k_s\|_{L^2}^2 + C_\varepsilon \|k_s\|_{L^2}^2 + \varepsilon \|k_{ss}\|_{L^2}^2 \\
&= C + C_\varepsilon \|k_s\|_{L^2}^2 + \varepsilon \|k_{ss}\|_{L^2}^2,
\end{aligned}$$

for any small value  $\varepsilon > 0$  and a suitable constant  $C_\varepsilon$ .

We deal now with the boundary term  $\sum_{i=1}^3 2(k^i)^3 k_s^i + \lambda^i (k_s^i)^2$ .

By the fact that  $k_s^i + \lambda^i k^i = k_s^j + \lambda^j k^j$ , for every pair  $i, j$ , it follows that  $(k_s + \lambda k)^2 \sum_{i=1}^3 \lambda^i = 0$ , hence,

$$\sum_{i=1}^3 \lambda^i (k_s^i)^2 = - \sum_{i=1}^3 (\lambda^i)^3 (k^i)^2 + 2 (\lambda^i)^2 k^i k_s^i,$$

then, we can write

$$\begin{aligned} \sum_{i=1}^3 2(k^i)^3 k_s^i + \lambda^i (k_s^i)^2 &= \sum_{i=1}^3 2(k^i)^3 k_s^i - (\lambda^i)^3 (k^i)^2 - 2(\lambda^i)^2 k^i k_s^i \\ &= \sum_{i=1}^3 2[(k^i)^3 - (\lambda^i)^2 k^i] k_s^i - \sum_{i=1}^3 (\lambda^i)^3 (k^i)^2 \\ &= 2(k_s + \lambda k) \sum_{i=1}^3 (k^i)^3 - (\lambda^i)^2 k^i + \sum_{i=1}^3 (\lambda^i)^3 (k^i)^2 - 2\lambda^i (k^i)^4. \end{aligned}$$

At the triple junction  $O^1$ , where the curves  $\gamma^1, \gamma^2$  and  $\gamma^5$  concur, there exists  $i \in \{1, 2\}$  such that  $|k^i(O^1)| \geq \frac{K}{2}$ , where  $K = \max_{j \in \{1, 2, 3\}} |k^j(O^1)|$ , hence at the 3-point  $O^1$

$$\begin{aligned} 2(k_s + \lambda k) \sum_{i=1}^3 (k^i)^3 - (\lambda^i)^2 k^i + \sum_{i=1}^3 (\lambda^i)^3 (k^i)^2 - 2\lambda^i (k^i)^4 \\ \leq CK^5 + C|k_s^i(O^1)|K^3 \\ \leq C|k^i(O^1)|^5 + C|k_s^i(O^1)||k^i(O^1)|^3 \\ \leq C\|k^i\|_{L^\infty(\gamma^i)}^5 + C\|k_s^i\|_{L^\infty(\gamma^i)}\|k^i\|_{L^\infty(\gamma^i)}^3. \end{aligned}$$

We estimate now  $C\|k\|_{L^\infty(\gamma^i)}^5 + C\|k_s\|_{L^\infty(\gamma^i)}\|k\|_{L^\infty(\gamma^i)}^3$  via the Gagliardo–Nirenberg interpolation inequalities in Proposition 5.7. Letting  $u = k^i$ ,  $p = +\infty$ ,  $m = 2$  and  $n = 0, 1$  in formula (5.5), we get

$$\begin{aligned} \|k^i\|_{L^\infty(\gamma^i)} &\leq C\|k_{ss}^i\|_{L^2(\gamma^i)}^{\frac{1}{4}}\|k^i\|_{L^2(\gamma^i)}^{\frac{3}{4}} + \frac{B}{L^{\frac{1}{2}}}\|k^i\|_{L^2(\gamma^i)} \leq C\|k_{ss}^i\|_{L^2(\gamma^i)}^{\frac{1}{4}} + C_L \\ \|k_s^i\|_{L^\infty(\gamma^i)} &\leq C\|k_{ss}^i\|_{L^2(\gamma^i)}^{\frac{3}{4}}\|k^i\|_{L^2(\gamma^i)}^{\frac{1}{4}} + \frac{B}{L^{\frac{3}{2}}}\|k^i\|_{L^2(\gamma^i)} \leq C\|k_{ss}^i\|_{L^2(\gamma^i)}^{\frac{3}{4}} + C_L, \end{aligned}$$

hence,

$$C\|k^i\|_{L^\infty(\gamma^i)}^5 + C\|k^i\|_{L^\infty(\gamma^i)}^3\|k_s^i\|_{L^\infty(\gamma^i)} \leq C\|k_{ss}^i\|_{L^2(\gamma^i)}^{\frac{5}{4}} + C\|k_{ss}^i\|_{L^2(\gamma^i)}^{\frac{3}{2}} + C_L \leq \varepsilon\|k_{ss}^i\|_{L^2(\gamma^i)}^2 + C_{L,\varepsilon}.$$

Thus, finally,

$$2(k_s + \lambda k) \sum_{i=1}^3 (k^i)^3 - (\lambda^i)^2 k^i + \sum_{i=1}^3 (\lambda^i)^3 (k^i)^2 - 2\lambda^i (k^i)^4 \leq \varepsilon\|k_{ss}^i\|_{L^2(\gamma^i)}^2 + C_{L,\varepsilon} \leq \varepsilon\|k_{ss}\|_{L^2}^2 + C_{L,\varepsilon}.$$

Coming back to computation (10.8), we have

$$\begin{aligned} &\partial_t \left( \int_{\mathbb{S}_t} k_s^2 ds - 2 \sum_{p=1}^2 \sum_{i=1}^3 \lambda^{pi} (k^{pi})^2 / 3 \right) \Big|_{\text{at the 3-point } O^p} \\ &\leq -2 \int_{\mathbb{S}_t} k_{ss}^2 ds + C\|k_s\|_{L^2}^2 + \varepsilon\|k_{ss}\|_{L^2}^2 + C_{L,\varepsilon} \\ &\leq -2 \int_{\mathbb{S}_t} k_{ss}^2 ds + C\|k_s\|_{L^2}^2 + 2\varepsilon\|k_{ss}\|_{L^2}^2 - C_{L,\varepsilon}\|k^i\|_{L^\infty(\gamma^i)}^3 + C_{L,\varepsilon} \\ &\leq C_{L,\varepsilon} \left( \int_{\mathbb{S}_t} k_s^2 ds - 2 \sum_{p=1}^2 \sum_{i=1}^3 \lambda^{pi} (k^{pi})^2 / 3 \right) \Big|_{\text{at the 3-point } O^p} + C_{L,\varepsilon}, \end{aligned}$$



where we chose  $\varepsilon < 1$ .

By Gronwall's Lemma, it follows that  $\|k_s\|_{L^2}^2 - 2 \sum_{p=1}^2 \sum_{i=1}^3 \lambda^{pi} (k^{pi})^2 / 3 \Big|_{\text{at the 3-point } O^p}$  is uniformly bounded, for  $t \in [0, T)$ , by a constant depending on  $L$  and its value on the initial network  $\mathbb{S}_0$ . Then, applying Young inequality to estimate (10.2) of Lemma 10.22, there holds

$$\|k\|_{L^\infty}^3 \leq C + C\|k\|_{L^2}^{3/2} \|k_s\|_{L^2}^{3/2} \leq C + C_\varepsilon \|k\|_{L^2}^6 + \varepsilon \|k_s\|_{L^2}^2 \leq C_\varepsilon + \varepsilon \|k_s\|_{L^2}^2,$$

as  $\|k\|_{L^2}$  is uniformly bounded in  $[0, T)$ . Choosing  $\varepsilon > 0$  small enough, we conclude that also  $\|k_s\|_{L^2}$  is uniformly bounded in  $[0, T)$ .  $\square$

*Proof of Proposition 10.21.* By the hypotheses, we can assume that the sequence of rescaled networks  $\mathbb{S}_{-1/(2+\delta)}^{\mu_i}$  converges in  $W_{\text{loc}}^{2,2}$ , as  $i \rightarrow \infty$ , to a standard cross (which has zero curvature), for some  $\delta > 0$  as small as we want.

Arguing as in the proof of Lemma 9.1, by means of Lemma 8.18, we can also assume that, for  $R > 0$  large enough, the sequence of rescaled flows  $\mathbb{S}_t^{\mu_i}$  converges smoothly and uniformly to the flow  $\mathbb{S}_t^\infty$ , given by the four halflines, in  $(B_{3R}(0) \setminus B_R(0)) \times [-1/2, 0)$ . Hence, there exists  $i_0 \in \mathbb{N}$  such that for every  $i \geq i_0$  the flow  $\mathbb{S}_t$  in the annulus  $B_{3R/\mu_i}(x_0) \setminus B_{R/\mu_i}(x_0)$  has equibounded curvature, no 3-points and an uniform bound from below on the lengths of the four curves, for  $t \in [T - \mu_i^{-2}/(2 + \delta), T)$ . Setting  $t_i = T - \mu_i^{-2}/(2 + \delta)$ , we have then a sequence of times  $t_i \rightarrow T$  such that, when  $i \geq i_0$ , the above conclusion holds for the flow  $\mathbb{S}_t$  in the annulus  $B_{3R\sqrt{2(T-t_i)}}(x_0) \setminus B_{R\sqrt{2(T-t_i)}}(x_0)$  and with  $t \in [t_i, T)$ , we can thus introduce four “artificial” moving boundary points  $P^r(t) \in \mathbb{S}_t$  with  $|P^r(t) - x_0| = 2R\sqrt{2(T-t_i)}$ , with  $r \in \{1, 2, 3, 4\}$  and  $t \in [t_i, T)$ , such that the estimates (5.1) are satisfied, that is, the hypotheses about the end-points  $P^i(t)$  of Lemmas 10.22, 10.23 and 10.24 hold.

As we the sequence of networks  $\mathbb{S}_{-1/(2+\delta)}^{\mu_i}$  converges in  $W_{\text{loc}}^{2,2}$  to a limit network with zero curvature, as  $i \rightarrow \infty$ , we have

$$\lim_{i \rightarrow \infty} \|\tilde{k}\|_{L^2(B_{3R}(0) \cap \mathbb{S}_{-1/(2+\delta)}^{\mu_i})} = 0, \quad \text{that is,} \quad \int_{B_{3R}(0) \cap \mathbb{S}_{-1/(2+\delta)}^{\mu_i}} \tilde{k}^2 d\sigma \leq \varepsilon_i,$$

for a sequence  $\varepsilon_i \rightarrow 0$  as  $i \rightarrow \infty$ . Rewriting this condition for the non-rescaled networks, we have

$$\int_{B_{3R\sqrt{2(T-t_i)}}(x_0) \cap \mathbb{S}_{t_i}} k^2 ds \leq \frac{\varepsilon_i}{\sqrt{2(T-t_i)}}.$$

Applying now Lemma 10.23 to the flow of networks  $\mathbb{S}_t$  in the ball  $B_{2R\sqrt{2(T-t_i)}}(x_0)$  in the time interval  $[t_i, T)$ , we have that  $\|k\|_{L^2(B_{2R\sqrt{2(T-t_i)}}(x_0) \cap \mathbb{S}_{t_i})}$  is uniformly bounded, up to time

$$T_i = t_i + \min \left\{ T, 1/8C \left( \|k\|_{L^2(B_{2R\sqrt{2(T-t_i)}}(x_0) \cap \mathbb{S}_{t_i})}^2 + 1 \right)^2 \right\}.$$

We want to see that actually  $T_i > T$  for  $i$  large enough, hence,  $\|k\|_{L^2(B_{2R}(x_0) \cap \mathbb{S}_t)}$  is uniformly bounded for  $t \in [t_i, T)$ . If this is not true, we have

$$\begin{aligned} T_i &= t_i + \frac{1}{8C \left( \|k\|_{L^2(B_{2R\sqrt{2(T-t_i)}}(x_0) \cap \mathbb{S}_{t_i})}^2 + 1 \right)^2} \\ &\geq t_i + \frac{1}{8C \left( \varepsilon_i / \sqrt{2(T-t_i)} + 1 \right)^2} \\ &= t_i + \frac{2(T-t_i)}{8C \left( \varepsilon_i + \sqrt{2(T-t_i)} \right)^2} \\ &= T + (2(T-t_i)) \left( \frac{2}{8C \left( \varepsilon_i + \sqrt{2(T-t_i)} \right)^2} - 1 \right), \end{aligned}$$

which is clearly larger than  $T$ , as  $\varepsilon_i \rightarrow 0$ , when  $i \rightarrow \infty$ .

Choosing then  $i_1 \geq i_0$  large enough, since  $\|k\|_{L^2(B_{2R}\sqrt{2(T-t_{i_1})}(x_0) \cap \mathbb{S}_t)}$  is uniformly bounded for all times  $t \in [t_{i_1}, T)$  and the length of the four curves that connect the junctions with the “artificial” boundary points  $P^r(t)$  are bounded below by a uniform constant, Lemma 10.24 applies, hence, thanks to Lemma 10.22, we have a uniform bound on  $\|k\|_{L^\infty(B_{2R}\sqrt{2(T-t_{i_1})}(x_0) \cap \mathbb{S}_t)}$  for  $t \in [0, T)$ .  $\square$

As we proved Proposition 10.21, Proposition 10.20 follows. An obvious consequence is that evolving trees do not develop this kind of singularities, hence, their curvature flow is smooth till a curve collapses with uniformly bounded curvature. Moreover, it is also easy to see that if no regions collapse, the network is a tree, for  $t$  close enough to  $T$ , around every point of  $\overline{\Omega}$ , so Proposition 10.20 applies globally.

**Corollary 10.25.** *If M1 holds and  $\mathbb{S}_0$  is a tree, the curvature of  $\mathbb{S}_t$  is uniformly bounded during the flow (hence, we are in the case of Corollary 10.17 in the previous section).*

Combining Propositions 10.18 and 10.20, we have the following local conclusion.

**Theorem 10.26.** *If M1 holds and  $\mathbb{S}_t$  is a tree in a neighborhood of  $x_0 \in \overline{\Omega}$ , for  $t$  close enough to  $T$ , either the flow  $\mathbb{S}_t$  is locally smooth or, up to reparametrization proportional to arclength, converge in  $C_{\text{loc}}^1$  locally around  $x_0$ , as  $t \rightarrow T$ , to some degenerate regular network  $\widehat{\mathbb{S}}_T$  whose non-degenerate curves form a  $C^1$  network  $\mathbb{S}_T$  with a possibly non-regular multi-point which is among the ones described in Proposition 10.14. Moreover, the curves of  $\mathbb{S}_T$  belong to  $C^1 \cap W^{2,\infty}$ , in a neighborhood of  $x_0$ , and are smooth outside the multi-point.*

*Remark 10.27.* By means of Lemma 10.24,  $\|k_s\|_{L^2}$  is locally uniformly bounded on  $[0, T)$ , which implies that the curves of  $\mathbb{S}_T$  are actually  $C^2$ , as well as the convergence of the non-collapsed curves of  $\mathbb{S}_t$ , as  $t \rightarrow T$ . By extending the estimates of Lemmas 10.22, 10.23 and 10.24 to the higher order derivatives of the curvature, one should actually get the smoothness of the curves and of the convergence.

Bounded curvature is not actually the case if some loops are present, indeed we have seen that a region bounded by less than six curves possibly collapses and in such case the curvature cannot stay bounded.

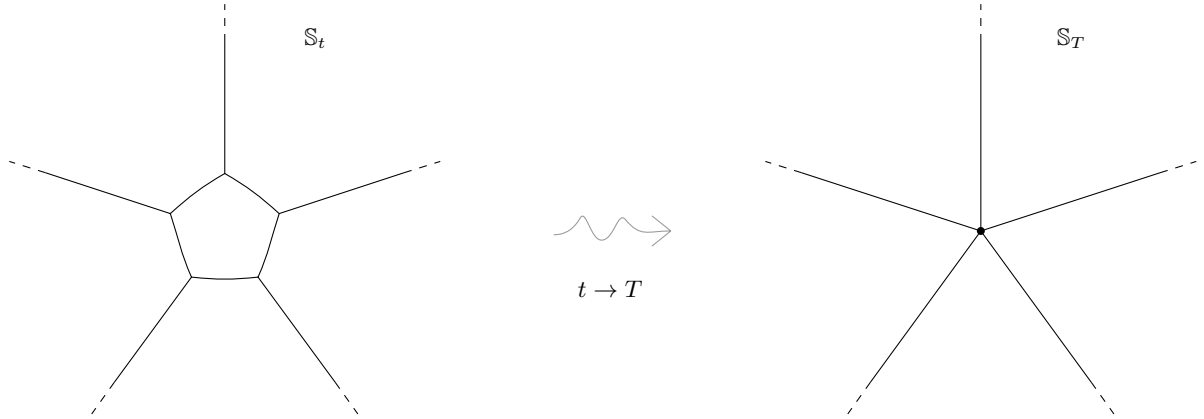


Figure 16: Homothetic collapse of a (symmetric) pentagonal region of  $\mathbb{S}_t$  (five-ray star).

Determining what asymptotically happens in details in the general case can be quite complicated, because of the difficulty in classifying the regular shrinkers with loops, anyway, some special cases with “few” triple junctions can be fully understood. We will show an example of this analysis in Section 15, considering networks with only one triple junction and we refer the reader to [62] for the complete study of the evolution of networks with only *two* 3-points. We underline that the interest in this latter case is due to the fact that the multiplicity-one conjecture holds for such networks (Corollary 14.11).

However, even if we cannot describe all the possible shrinkers  $\mathbb{S}_{-1/2}^\infty$  or  $\widetilde{\mathbb{S}}_\infty$ , arising respectively from the parabolic or Huisken’s rescaling procedure at the singular time  $T < +\infty$ , we can get enough information in order to restart the flow by means of Theorem 11.1 in the next section (actually by its

extension discussed in Remark 11.20). The point is to connect the information on the possible blow-up limit networks  $\tilde{S}_\infty$  to the existence and the structure of a network  $S_T$  which is the limit of  $S_t$ , as  $t \rightarrow T$ .

We recall that assuming the multiplicity-one conjecture, by Lemma 10.7, there exist the limits  $x_i = \lim_{t \rightarrow T} O^i(t)$ , for  $i \in \{1, 2, \dots, m\}$  and correspond to the (finitely many) points in  $\Omega$  where  $\hat{\Theta}(x_0) > 1$  and to the end-points of  $S_t$  such that the curve getting there “collapses” as  $t \rightarrow T$ .

We first discuss what happens around an end-point  $P^r$  of the network  $S_t$  if  $x_i = P^r$  for some (possibly more than one)  $i \in \{1, 2, \dots, m\}$ . As before, we consider the network  $\mathbb{H}_t^r$ , obtained by the union of  $S_t$  with its reflection with respect to the point  $P^r$  (see just before Section 10.1). If  $\Omega$  is strictly convex, by Proposition 8.12, every blow-up limit network  $\tilde{\mathbb{H}}_\infty^r$ , obtained rescaling around the end-point  $P^r$ , must be symmetric and contained in the union of two cones for the origin of  $\mathbb{R}^2$ . Then, by an argument similar to the one in the proof of Lemma 8.10, either  $\tilde{\mathbb{H}}_\infty^r$  is a tree, or it contains a loop around the origin, which is clearly impossible by such property. Hence, we conclude that  $\tilde{\mathbb{H}}_\infty^r$  is a tree and the same the blow-up limit network  $\tilde{S}_\infty$ , which means that we are in the previous case, considered in Proposition 10.20, in particular, the curvature is locally bounded.

Then, by Proposition 10.18 (and 10.14), we have a complete description of the behavior of  $S_t$  locally around its end-point, as  $t \rightarrow T$ .

**Theorem 10.28.** *If M1 holds and the open set  $\Omega$  is strictly convex, then in a neighborhood of its fixed end-points on  $\partial\Omega$ , the evolving regular network  $S_t$  is a tree, for  $t$  close enough to  $T$ , and its curvature is uniformly locally bounded during the flow. Hence, around any end-point  $P^r$  either the flow is smooth, or the curve of  $S_t$  getting to  $P^r$  collapses and the network  $S_t$  locally converges in  $C^1$ , as  $t \rightarrow T$ , to two concurring curves at such end-point forming an angle of 120 degrees, as in the right side of Figure 14.*

*Remark 10.29.* We remark that the strictly convexity hypothesis on  $\Omega$  can actually be weakened asking that  $\Omega$  is convex and that there do not exist three aligned end-points of the initial network  $S_0$  on  $\partial\Omega$ .

We now deal with the situation of a point  $x_0 = \lim_{t \rightarrow T} O^i(t)$ , for some  $i \in \{1, 2, \dots, m\}$ , with  $x_0 \in \Omega$ . Assuming that around  $x_0 \in \Omega$  the network is not a tree for  $t$  close enough to  $T$  (which would imply that the curvature is locally bounded, by Proposition 10.20), there must be at least one bounded region of  $S_t$  collapsing to  $x_0$  at the singular time. By the estimates in Section 8.2, then the area  $A(t)$  of any such region must satisfy  $A(t) = C(T - t)$ , for some constant  $C$  depending on the number of its edges. Hence, all the rescaled networks  $\tilde{S}_{x_0, t}$  must contain the rescalings of such regions that will have a respective constant area. These rescaled regions cannot “go all to infinity” and disappear in the blow-up limit network  $\tilde{S}_\infty$ , along any converging sequence  $\tilde{S}_{x_0, t_j} \rightarrow \tilde{S}_\infty$ , otherwise Lemma 8.9 would apply and we could repeat the argument of the proof of Proposition 10.20, concluding that the curvature is uniformly bounded around  $x_0$ .

If the full rescaled family of networks  $\tilde{S}_{x_0, t}$  converges to  $\tilde{S}_\infty$  (for instance, if the uniqueness assumption U in Problem 8.22 holds), we separate  $\tilde{S}_\infty$  in two parts:

- a compact subnetwork  $\tilde{M}_\infty$  of  $\tilde{S}_\infty$ , given by the union of the cores and the bounded curves (which are pieces of Abresch–Langer curves or straight segments passing by the origin of  $\mathbb{R}^2$ ),
- the union  $\tilde{N}_\infty = \tilde{S}_\infty \setminus \tilde{M}_\infty$  of the unbounded curves of  $\tilde{S}_\infty$ , which must be halflines “pointing” towards the origin (but not necessarily containing it), by Remark 8.8.

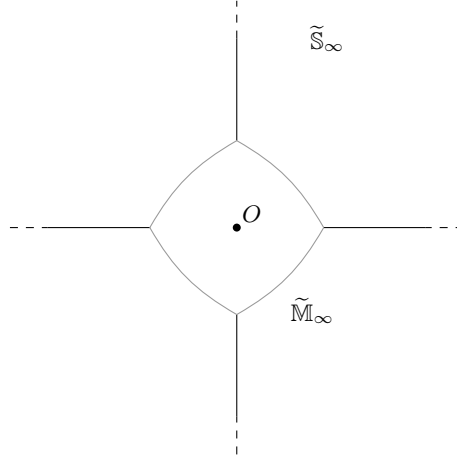


Figure 17: The subnetwork  $\tilde{\mathbb{M}}_\infty$  (in gray) of a 4-symmetric regular shrinker  $\tilde{\mathbb{S}}_\infty$  (four-ray star)

Then, by rescaling-back (dynamically contracting) the flow  $\tilde{\mathbb{S}}_{x_0,t} \rightarrow \tilde{\mathbb{S}}_\infty$ , by the uniqueness assumption, the subnetwork  $\mathbb{M}_t$  of  $\mathbb{S}_t$  corresponding to the compact subnetwork of  $\tilde{\mathbb{S}}_{x_0,t}$  converging to  $\tilde{\mathbb{M}}_\infty$ , is contained in the ball  $B_{C\sqrt{2(T-t)}/2}(x_0)$  for every  $t \in [0, T)$ , for some constant  $C$  independent of  $t$  (dependent on  $\tilde{\mathbb{M}}_\infty$ ). In particular,  $\mathbb{M}_t$  completely collapses to the point  $x_0$ , “disappearing” in the limit, as  $t \rightarrow T$ . We want now to describe the local behavior of the rest  $\mathbb{N}_t$  of the network  $\mathbb{S}_t$  (corresponding to the union of the curves of  $\tilde{\mathbb{S}}_{x_0,t}$  neither collapsing, nor entirely going to infinity, converging to the halflines of  $\tilde{\mathbb{S}}_\infty$ ), around the point  $x_0$ , as  $t \rightarrow T$ .

**Proposition 10.30.** *If the multiplicity-one conjecture and the above uniqueness assumption of the blow-up limit shrinker  $\tilde{\mathbb{S}}_\infty$  hold, then, as  $t \rightarrow T$ , the family  $\gamma_t^i$  of curves of  $\mathbb{N}_t$  converges in  $C^1(U)$  and in  $C^\infty(U \setminus \{x_0\})$ , where  $U$  is a neighborhood of  $x_0$ , as  $t \rightarrow T$ , to an embedded, possibly non-regular network  $\mathbb{S}_T$ , composed of  $C^1$  curves  $\gamma_T^i$  concurring at  $x_0$ .*

*The directions of the halflines of  $\tilde{\mathbb{S}}_\infty$  coincide with the inner unit tangent vectors of the limit curves  $\gamma_T^i$  at  $x_0$ , hence, these latter are all distinct.*

*Moreover, the curvature of every curve  $\gamma_T^i$  is of order  $o(1/r)$ , as  $r \rightarrow 0$ , where  $r$  is the distance from the multi-point  $x_0 \in \mathbb{S}_T$ .*

*Proof.* Since rescaling the evolving networks  $\mathbb{S}_t$  the inner unit tangent vectors at the end-points of the curves in  $\mathbb{N}_t$  do not change and  $\tilde{\mathbb{N}}_{x_0,t} \rightarrow \tilde{\mathbb{N}}_\infty$ , the inner unit tangent vectors of the set of curves  $\gamma_t^i$  converge to the unit vectors generating the halflines of  $\tilde{\mathbb{S}}_\infty$ . More precisely, if the sequence of rescalings  $\tilde{\gamma}_{x_0,t}^i$  of a curve  $\gamma_t^i \in \mathbb{N}_t$  converges in  $C_{\text{loc}}^1$  to a halfline  $H^i \subset \tilde{\mathbb{N}}_\infty$ , the inner unit tangent vectors at the end-point of  $\gamma_t^i$  converge to the unit vector generating  $H^i$ , as  $t \rightarrow T$ .

As, by Lemma 10.7 and the collapse of the subnetwork  $\mathbb{M}_t$ , there is a neighborhood  $U$  of  $x_0$ , such that in  $U \setminus \{x_0\}$ , for  $t$  close enough to  $T$ , there are no triple junctions, by Lemma 8.18, the networks  $\mathbb{S}_t$  converge in  $C_{\text{loc}}^\infty(U \setminus \{x_0\})$  to a smooth network  $\mathbb{S}_T$  composed of smooth curves  $\gamma_T^i$  with an end-point at  $x_0$ .

We notice that the smoothness of  $\mathbb{S}_T$  and of  $\gamma_T^i$  holds in  $U \setminus \{x_0\}$ , not in the whole  $U$ . We want to show that these curves are actually  $C^1$  in  $U$ , that is, till the point  $x_0$  and that their curvature is of order  $o(1/r)$ , where  $r$  is the distance from  $x_0$ .

We consider one of the curves of  $\mathbb{N}_t$  (dropping the superscript by simplicity, from now on)  $\gamma_t$ , which converges (possibly, after reparametrization), as  $t \rightarrow T$ , to a limit  $C^0$  curve  $\gamma_T$  and such convergence is also in  $C_{\text{loc}}^\infty(U \setminus \{x_0\})$ .

As the full rescaled sequence  $\tilde{\mathbb{S}}_t$  converges to the blow-up limit  $\tilde{\mathbb{S}}_\infty$ , as  $t \rightarrow +\infty$ , also the full sequence of parabolically rescaled flows  $\mathbb{S}_t^\mu$  converges in  $C_{\text{loc}}^1$  for every  $t \in (-\infty, 0)$ , as  $\mu \rightarrow +\infty$ , to the limit self-similarly shrinking flow  $\mathbb{S}_t^\infty = \sqrt{-2t} \tilde{\mathbb{S}}_\infty$  (see Remark 7.4). Then, the curves  $\gamma_t^\mu$ , which are the parabolic rescalings of the curves  $\gamma_t$  converge to the halfline  $H$ , as  $\mu \rightarrow +\infty$ . We choose  $t_0 < 0$  and  $\mu_0 > 0$  such that the parabolic rescalings  $\mathbb{M}_t^\mu$  of the subnetwork  $\mathbb{M}_t$  of  $\mathbb{S}_t$  are contained in  $B_{1/2}(0)$ , for every  $\mu > \mu_0$  and  $t \in (t_0, 0)$ . Then, the rescaled curves  $\gamma_t^\mu$  smoothly converge (by Lemma 8.18), as

$\mu \rightarrow +\infty$ , to the halfline  $H$  (which has zero curvature) in  $B_4(0) \setminus B_1(0)$ , for every  $t \in [t_0, 0)$ . Moreover, repeating the above argument, we have that, as  $t \rightarrow 0$ , the curves  $\gamma_t^\mu$  locally smoothly converge in  $B_4(0) \setminus \{0\}$  to some limit curves  $\gamma_0^\mu$ , smooth in  $B_3(0) \setminus \{0\}$ , for every fixed  $\mu > \mu_0$ .

We are now going to apply the following special case of the *pseudolocality theorem* for mean curvature flow (see [47, Theorem 1.5]) and the subsequent remark.

**Theorem 10.31.** *Let  $\gamma_t$ , for  $t \in [0, T)$ , be a smooth curvature flow of an embedded curve in  $\mathbb{R}^2$  with bounded length ratios by a constant  $D$  (see Definition 8.14) and let*

$$Q_r(x_0, y_0) = \{(x, y) \in \mathbb{R}^2 \mid |x - x_0| < r, |y - y_0| < r\}.$$

*Then, for any  $\varepsilon > 0$ , there exists  $\eta \in (0, \varepsilon)$  and  $\delta \in (0, 1)$ , depending only on  $\varepsilon$  and  $D$ , such that if  $(x_0, y_0) \in \gamma_0$  and  $\gamma_0 \cap Q_1(x_0, y_0)$  can be written as the graph of a function  $u : (x_0 - 1, x_0 + 1) \rightarrow \mathbb{R}$  with Lipschitz constant less than  $\eta$ , then*

$$\gamma_t \cap Q_\delta(x_0, y_0), \quad \text{for every } t \in [0, \delta^2) \cap [0, T),$$

*is a graph over  $(x_0 - \delta, x_0 + \delta)$  of a function with Lipschitz constant less than  $\varepsilon$  and “height” bounded by  $\varepsilon\delta$ .*

**Remark 10.32.** Then, the local estimates of Ecker and Huisken [26] imply that, for every  $\lambda > 0$  there is a constant  $\sigma = \sigma(\delta, \varepsilon, \lambda) > 0$  and a constant  $\eta = \eta(\delta, \varepsilon, \lambda) > 0$  such that if the curvature of  $\gamma_0 \cap Q_\delta(x_0, y_0)$  is bounded by  $\sigma$ , then the curvature of  $\gamma_t \cap Q_{\delta/2}(x_0, y_0)$  is bounded by  $\lambda$ , for every  $t \in [0, \eta) \cap [0, T)$ .

By a rotation, we can assume that  $H = \{(x, 0) \mid x \geq a\}$  and let  $\overline{H} = \{(x, 0) \mid x \geq 0\}$ . Taken any  $\varepsilon > 0$ , let  $\eta$  and  $\delta$  be given by this theorem, we consider  $t_1 \in (t_0, 0)$  such that  $t_1 + \delta^2/8 > 0$ , then if  $\mu$  is large enough, say larger than some  $\mu_1 > 0$ , the curve  $\gamma_{t_1}^\mu$  in  $B_3(0) \setminus B_1(0)$  is a graph of a function  $u$  over the interval  $[1, 3] \times \{0\} \subset \overline{H}$  (with a small “error” at the borders), with gradient smaller than  $\eta > 0$ . Hence, its evolution in the smaller annulus  $B_{2+\delta}(0) \setminus B_{2-\delta}(0)$  is still a graph over  $\overline{H}$  of a function with gradient smaller than  $\varepsilon$ , for every  $t \in [t_1, \min\{t_1 + \delta^2, 0\})$ , hence for every  $t \in [t_1, 0)$ , by the assumption on  $t_1$ . Notice that, it follows that also  $\gamma_0^\mu$  in  $B_{2+\delta}(0) \setminus B_{2-\delta}(0)$  is a graph of a function over  $\overline{H}$  with gradient smaller than  $\varepsilon$ , when  $\mu > \mu_1$ .

Rescaling back, since the  $C^1$ -norm is scaling invariant, we see that  $\gamma_t$ , for  $t \in [T + \mu^{-2}t_1, T]$ , can be written as a graph with  $C^1$ -norm less than  $\varepsilon$  over  $x_0 + \overline{H}$  in  $B_{(2+\delta)/\mu}(x_0) \setminus B_{(2-\delta)/\mu}(x_0)$ , for every  $\mu > \mu_1$ . Hence, this conclusion holds for every pair  $(\gamma_t, t)$  in

$$\bigcup_{\mu > \mu_2} (B_{(2+\delta)/\mu}(x_0) \setminus B_{(2-\delta)/\mu}(x_0)) \times [T + \mu^{-2}t_1, T] \subset \mathbb{R}^2 \times [0, T],$$

for every  $\mu_2 \geq \mu_1$ , and this union contains the set

$$\mathcal{A} = B_{(2+\delta)/\mu_2}(x_0) \times [T + \mu_2^{-2}t_1, T] \setminus \left\{ (x, t) \in \mathbb{R}^2 \times [0, T] \mid |x - x_0| \leq \frac{2 - \delta}{\sqrt{-2t_1}} \sqrt{2(T - t)} \right\}.$$

Choosing now  $\mu_2 \geq \mu_1$  large enough, we know that there exists some  $t_2 > t_1$  such that for every  $t > t_2$ , the rescaled curves  $\gamma_t^{\mu_2}$  can be written as graphs with  $C^1$ -norm less than  $\varepsilon$  over  $\overline{H}$  in the ball centered at the origin with radius  $2\frac{2-\delta}{\sqrt{-2t_1}}$ . That is, for  $t \in [T + \mu_2^{-2}t_2, T]$ , the curve  $\gamma_t$  can be written as a graph with  $C^1$ -norm less than  $\varepsilon$  over  $x_0 + \overline{H}$  in the ball of center  $x_0$  and radius  $2\frac{2-\delta}{\sqrt{-2t_1}}\sqrt{2(T - t)}$ , hence, for every  $(\gamma_t, t)$  in

$$\mathcal{B} = \left\{ (x, t) \in \mathbb{R}^2 \times [T + \mu_2^{-2}t_2, T] \mid |x - x_0| < 2\frac{2 - \delta}{\sqrt{-2t_1}} \sqrt{2(T - t)} \right\},$$

The union of the sets  $\mathcal{A}$  and  $\mathcal{B}$  clearly contains the set

$$B_{(2+\delta)/\mu_2}(x_0) \times [T + \mu_2^{-2}t_2, T] \setminus \{(x_0, T)\},$$

hence, in other words, for every  $\varepsilon > 0$  there exists a radius  $R_\varepsilon > 0$  and a time  $t_\varepsilon < T$  such that the curve  $\gamma_t$  in the ball  $B_{R_\varepsilon}(x_0)$  can be written as a graph with  $C^1$ -norm less than  $\varepsilon$ , for every  $t \in [t_\varepsilon, T)$ . Moreover, this also holds for the limit curve  $\gamma_T$  on the union

$$\bigcup_{\mu > \mu_2} (B_{(2+\delta)/\mu}(x_0) \setminus B_{(2-\delta)/\mu}(x_0)) = B_{(2+\delta)/\mu_2}(x_0) \setminus \{x_0\}.$$

This fact, recalling that the inner unit tangent vector of the curve  $\gamma_t$  at its end-point (the one going to  $x_0$ ) converges to the direction of  $H$ , as  $t \rightarrow T$ , clearly shows that, locally around  $x_0$ , we can write  $\gamma_T$  as a graph of a function over  $x_0 + \overline{H}$  whose  $C^1$ -norm decays like  $o(1)$ , as the distance from  $x_0$  goes to zero. In particular, we conclude that all the curves  $\gamma_T^i$ , hence the limit network  $\mathbb{S}_T$ , are of class  $C^1$  and that all the sequences of curves  $\gamma_t^i$  converge in  $C^1$  to  $\gamma_T^i$  (possibly after reparametrization in arclength).

Arguing similarly for the curvature by means of Remark 10.32, we have that the curvature of the curve  $\gamma_0^\mu$  in  $B_{2+\delta/2}(0) \setminus B_{2-\delta/2}(0)$  is smaller than any  $\lambda > 0$ , if we choose  $\mu$  large enough, say  $\mu > \mu_3 \geq \mu_2$ . It follows, rescaling back, that

$$\mu^{-2} \sup_{\mathbb{S}_T \cap B_{(2+\delta/2)/\mu}(x_0) \setminus B_{(2-\delta/2)/\mu}(x_0)} k^2 < \lambda,$$

for every  $\mu > \mu_3$ . This implies that the curvature of  $\mathbb{S}_T$  is of order  $o(1/r)$ , as  $r \rightarrow 0$ , where  $r$  is the distance from the multi-point  $x_0 \in \mathbb{S}_T$ .

Finally,  $\mathbb{S}_T$  cannot have two concurring curve at a multi-point with the same unit tangent, since this would imply that the limit shrinker  $\tilde{\mathbb{S}}_\infty$  had halflines of multiplicity larger than one.  $\square$

It follows by this proposition that the networks  $\mathbb{S}_t$  converge in  $C^1(U)$  to a degenerate regular network  $\widehat{\mathbb{S}}_T$  having  $\mathbb{S}_T$  as non-collapsed part, with underlying graph homeomorphic to  $\mathbb{S}_t$  and core given by the collapsing subnetwork  $\mathbb{M}_t$ .

*Remark 10.33.* Notice that the limit Gaussian density  $\widehat{\Theta}(x_0) = \widehat{\Theta}(x_0, T)$  (see Definition 7.3) at  $x_0$  (and time  $T$ ) of the flow  $\mathbb{S}_t$  is the Gaussian density of the blow-up limit shrinker  $\tilde{\mathbb{S}}_\infty = \mathbb{S}_{-1/2}^\infty$  and can be different from the number of curves of  $\mathbb{S}_T$  concurring at  $x_0$ , divided by two. This does not happen when the network  $\mathbb{S}_t$  is a tree in a neighborhood of  $x_0$ , for  $t$  close enough to  $T$ , and the singularity is given by the collapsing of a single curve producing a 4-point with angles of 60/120 degrees between the four concurring curves, as described in Proposition 10.14 (after applying Proposition 10.20), in such case the blow-up limit shrinker is a standard cross and the limit Gaussian density  $\widehat{\Theta}(x_0, T)$  is clearly equal to two.

We actually expect that the curvature of the curves in  $\mathbb{N}_t$  and of  $\mathbb{S}_T$  is bounded, not only of order  $o(1/r)$ , close to the non-regular multi-points.

#### Open Problem 10.34.

- The curvature of  $\mathbb{S}_T$  is bounded?
- The curvature of the subnetwork  $\mathbb{N}_t$  is locally uniformly bounded around  $x_0$ , as  $t \rightarrow T$ ?

We can finally describe the local behavior of the whole network  $\mathbb{S}_t$ , as  $t \rightarrow T$ , around a point  $x_0 \in \Omega$  where  $\mathbb{S}_t$  is not a tree for  $t$  close enough to  $T$ .

**Theorem 10.35.** *Let  $x_i = \lim_{t \rightarrow T} O^i(t) \in \overline{\Omega}$ , for  $i \in \{1, 2, \dots, m\}$ , and let  $x_0$  one of such points such that  $x_0 \in \Omega$  and the blow-up limit at  $x_0$ , as  $t \rightarrow T$ , is not a line, a standard triod or a standard cross. Then, under the uniqueness assumption **U** and the multiplicity-one conjecture **M1**, there exists a  $C^1$ , possibly non-regular network  $\mathbb{S}_T$  in a neighborhood  $U$  of  $x_0$ , which is smooth in  $U \setminus \{x_0\}$  and whose curvature is of order  $o(1/r)$ , as  $r \rightarrow 0$ , where  $r$  is the distance from  $x_0$ , such that*

$$\mathbb{N}_t \rightarrow \mathbb{S}_T \text{ in } C_{\text{loc}}^1(U) \quad \text{and} \quad \mathbb{S}_t \rightarrow \mathbb{S}_T \text{ in } C_{\text{loc}}^\infty(U \setminus \{x_0\}),$$

where  $\mathbb{N}_t$  is the subnetwork of the non-collapsing curves of  $\mathbb{S}_t$ .

Moreover, at the multi-point  $x_0$  of  $\mathbb{S}_T$  any two concurring curves cannot have the same exterior unit tangent vectors.

The network  $\mathbb{S}_T$  is the non-collapsed part of a  $C^1$  degenerate regular network  $\widehat{\mathbb{S}}_T$  in  $U$  with underlying graph homeomorphic to  $\mathbb{S}_t$  and core given by the collapsed subnetwork  $\mathbb{M}_t$ , which is the  $C^1$ -limit of  $\mathbb{S}_t$ , as  $t \rightarrow T$ .

*Remark 10.36.* It is easy to see that, thanks to the uniformly bounded length ratios of  $\mathbb{S}_t$ , the one-dimensional Hausdorff measures associated to  $\mathbb{S}_t$  weakly-converge (as measures) to the one-dimensional Hausdorff measure associated to  $\mathbb{S}_T$  (see Remark 8.5).



## 10.4 Continuing the flow

We resume in the following two theorems the behavior of the evolving regular network at a singular time, worked out in the previous sections, assuming the multiplicity-one conjecture 10.1 and the uniqueness assumption 8.22.

**Theorem 10.37.** *If M1 is true and the uniqueness assumption U holds, then the (possibly simultaneous) singularities of the curvature flow of a regular network  $\mathbb{S}_t$  in a strictly convex, open subset  $\Omega \subset \mathbb{R}^2$  are given by:*

- *the collapse with bounded curvature of the “boundary curve”, locally around a fixed end-point on  $\partial\Omega$  where a singularity occurs (recall that regions cannot collapse to boundary points); indeed, around any end-point  $P^r$  either the flow is smooth, or the curve of  $\mathbb{S}_t$  getting to  $P^r$  collapses letting two concurring curves forming an angle of 120 degrees;*
- *the collapse with bounded curvature of a curve with the formation of a 4-point, locally around a point of  $\Omega$  where a singularity occurs; in this case the network is locally a tree, as  $t \rightarrow T$ ;*
- *the collapse to a point of  $\Omega$  of a group of bounded regions (each one of them with less than six boundary curves), producing a possibly non-regular multi-point.*

*If  $\{y_1, y_2, \dots, y_n, z_1, z_2, \dots, z_m\}$  are the points of  $\Omega$  where such singularities occur (which are a subset of the limits, as  $t \rightarrow T$ , of the 3-points of  $\mathbb{S}_t$ ), where we denoted with  $y_i$  the “cross” or “boundary” singularities (notice that the former must have  $\hat{\Theta}(y_i) = 2$ ) and with  $z_j$  the other singularities, where the network  $\mathbb{S}_t$  is not locally a tree, for  $t$  close enough to  $T$ , then there exists a possibly non-regular  $C^1$  limit network  $\mathbb{S}_T$  such that:*

- *every two concurring curves at a multi-point of  $\mathbb{S}_T$  have distinct exterior unit tangent vectors;*
- *the network  $\mathbb{S}_T$  is smooth in  $\overline{\Omega} \setminus \{y_1, y_2, \dots, y_n, z_1, z_2, \dots, z_m\}$ ;*
- *the curvature of  $\mathbb{S}_T$  is of order  $o(1/r)$ , as  $r \rightarrow 0$ , where  $r$  is the distance from the set of points  $\{z_i\}$ ;*
- *the network  $\mathbb{S}_t$  converges locally smoothly to  $\mathbb{S}_T$ , as  $t \rightarrow T$ , in  $\overline{\Omega} \setminus \{y_1, y_2, \dots, y_n, z_1, z_2, \dots, z_m\}$ ;*
- *the network  $\mathbb{S}_t$  converges locally in  $C^1$  to  $\mathbb{S}_T$ , as  $t \rightarrow T$ , in  $\overline{\Omega} \setminus \{z_1, z_2, \dots, z_m\}$ ;*
- *the non-collapsing subnetwork  $\mathbb{N}_t$  of  $\mathbb{S}_t$  converges locally in  $C^1$  to  $\mathbb{S}_T$ , as  $t \rightarrow T$ , in  $\overline{\Omega}$ ;*
- *the network  $\mathbb{S}_t$  converges locally in  $C^1$  to  $\hat{\mathbb{S}}_T$ , as  $t \rightarrow T$ , in  $\overline{\Omega}$ , where  $\hat{\mathbb{S}}_T$  is a degenerate regular network having  $\mathbb{S}_T$  as non-collapsed part.*

The case of a tree is special (for instance, the uniqueness assumption U is not needed in this case).

**Theorem 10.38.** *If M1 is true and the evolving regular network  $\mathbb{S}_t$  is a tree (or no regions are collapsing, as  $t \rightarrow T$ ), then the only possible singularities are given by either the collapses of a curve in the interior of  $\Omega$ , with the two triple junctions at the end-points of the curve going to collide, producing a 4-point where the four concurring curves have pairwise opposite exterior unit tangent vectors and form angles of 120/60 degrees between them, or (possibly simultaneously) the collapse of a curve to an end-point of the network, letting two curves concurring at such end-point forming an angle of 120 degrees between them.*

*The network  $\mathbb{S}_t$  converges to a limit network  $\mathbb{S}_T$  in  $C^2(\overline{\Omega})$ , with uniformly bounded curvature, as  $t \rightarrow T$ . Moreover, outside the 4-points and the “collapsing” points on  $\partial\Omega$ , the network  $\mathbb{S}_T$  is smooth and the convergence  $\mathbb{S}_t \rightarrow \mathbb{S}_T$  is also smooth.*

The next step, after this description, is to understand how the flow can continue after a singular time. There are clear situations where the flow simply ends, for instance if all the network collapses to a single point (inside  $\Omega$  since, by Theorem 10.28, this cannot happen to an end-point on the boundary, which means that the network  $\mathbb{S}_t$  is actually without end-points at all), like a circle shrinks down to a point in the evolution of a closed embedded single curve, see, for instance, the following example.



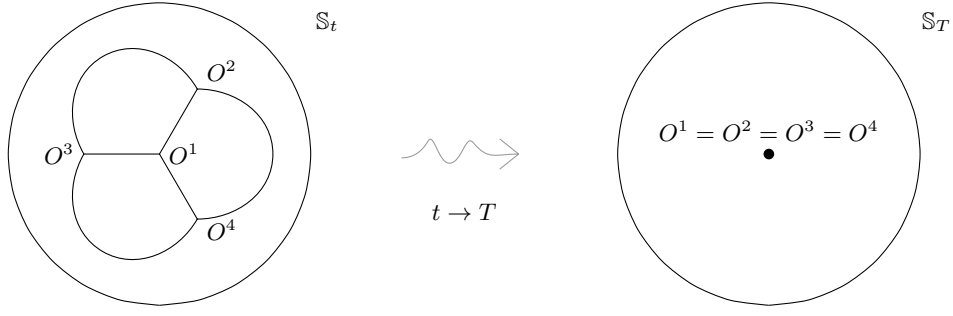


Figure 18: A Mercedes–Benz shrinker collapsing to a single point (see the Appendix).

In other situations, how the flow should continue is easy to guess or define, for instance if a part of the network collapses forming a 2–point, that can be also seen simply as an interior corner point of a single curve (see the following figure).

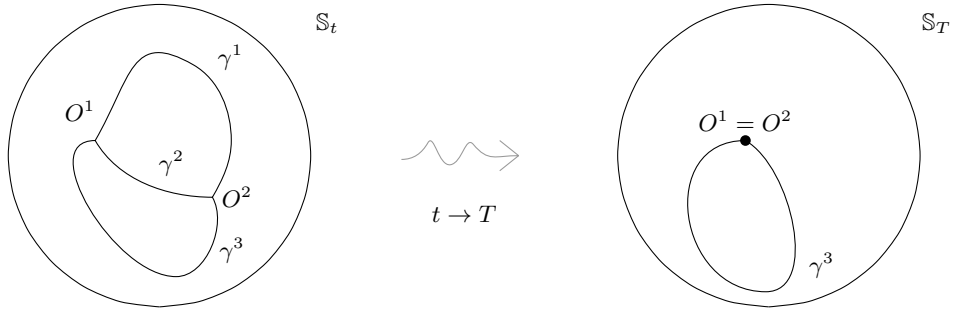


Figure 19: Collapse of both the curves  $\gamma^1, \gamma^2$  and the region they enclose to the point  $O^1 = O^2$ , leaving a closed curve  $\gamma^3$  with a corner at  $O^1 = O^2$  of 120 degrees.

Here, we can restart the network, by means of the work of Angenent [6–8] where the evolution of curves with corners is also treated (see Remark 2.2). In general, one would need an analogue of the small time existence Theorem 4.7 or 4.18, for networks with 2–points or with curves with corners. This will be actually a particular case of Theorem 11.1 in the next section (see the beginning of Section 11.3).

Instead, a situation that really needs a “decision” about whether and how the flow should continue after the singularity, is depicted in the following figures.

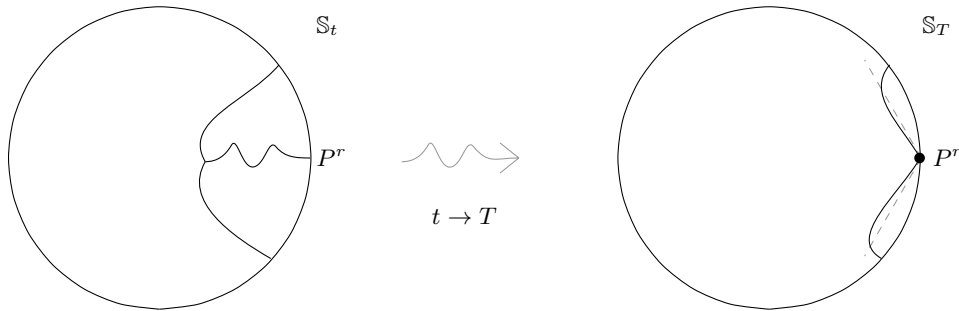


Figure 20: A limit network with two curves arriving at the same end–point on  $\partial\Omega$ .

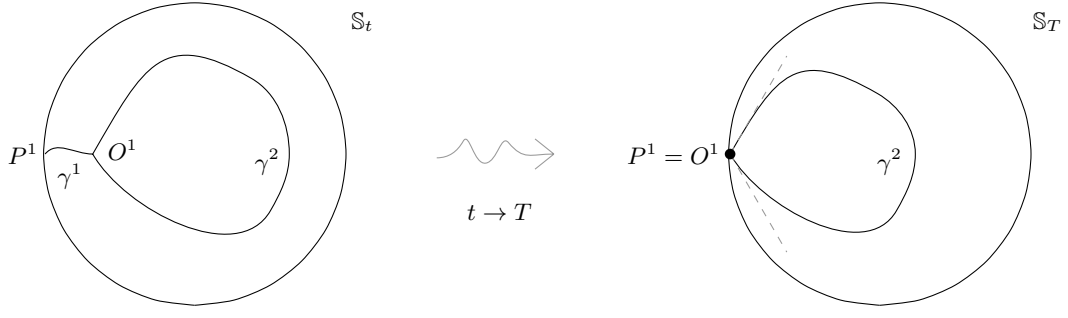


Figure 21: Collapse of the curve  $\gamma^1$  leaving a closed curve  $\gamma^2$  with an angle of 120 degrees at an end-point.

One can decide that the flow stops at  $t = T$  or that the curves become extremal curves of a new network that must have, for every  $t > T$ , a fixed end in the end-point  $P^r$  (this would require some analogues of the small time existence Theorems 4.7 and 4.18 for this class of non-regular networks, which are actually possible to be worked out). Anyway, the subsequent analysis becomes more troublesome because of such concurrency at the same end-point, indeed, it should be allowed that, at some time  $t > T$ , a new curve and a new 3-point “emerges” from such end-point (it would be needed a “boundary” extension of Theorem 11.1 in the next section).

Another situation that also needs a decision, but in this case easier, is described in the following figures.

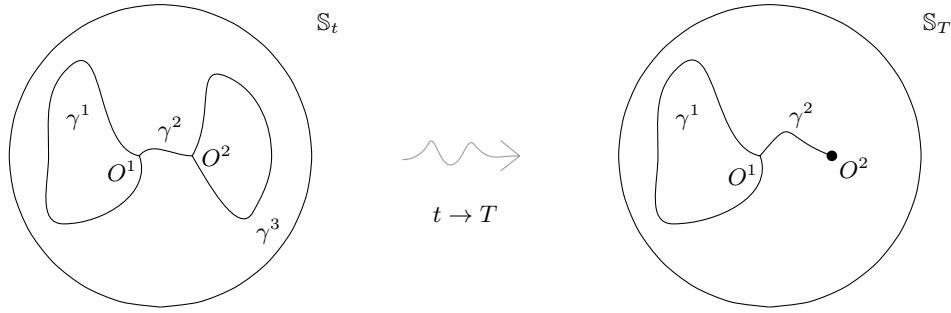


Figure 22: Collapse of the curves  $\gamma^3$  and the region enclosed to the point  $O^3$  leaving a curve  $\gamma^2$  with a 1-point as an end-point.

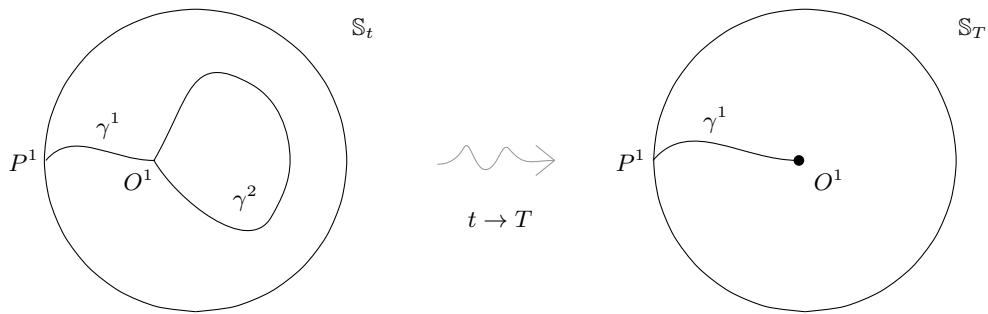


Figure 23: Collapse of the curves  $\gamma^2$  and the region enclosed to the point  $O^1$  leaving a curve  $\gamma^1$  with a 1-point as an end-point.

If the limit network  $\mathbb{S}_T$  contains a curve (or curves) which ends in a 1-point, it is actually natural to impose that such curve vanishes for every future time, so considering only the evolution of the network

of the rest of the network  $\mathbb{S}_T$  according to the above discussion (cutting away such a curve will produce a 2-point or the empty set, in the figures above, for instance).

Theorem 11.1 in the next section will give a way to restart the flow in the “nice” singularity situation described in Theorem 10.38, when a single curve, with bounded curvature collapses to an interior point of  $\Omega$  forming a non-regular network with a 4-point where the four concurring curves have opposite exterior unit tangent vectors in pairs and form angles of 120/60 degrees between them.

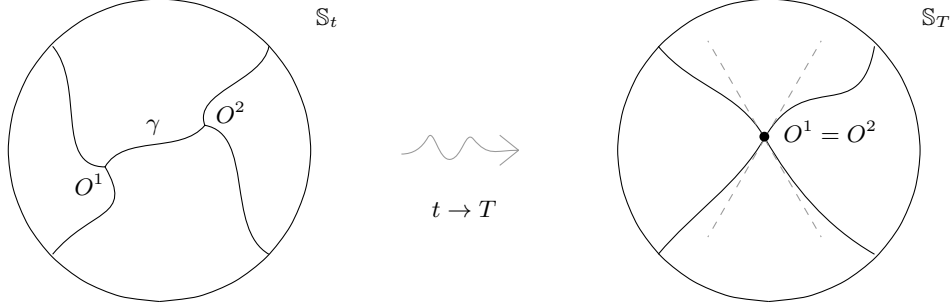


Figure 24: A limit “nice” collapse of a single curve  $\gamma$  producing a non-regular network  $\mathbb{S}_T$ .

Finally, if we are in the situation of a non-regular limit network  $\mathbb{S}_T$  described by Theorem 10.37, after the collapse of a region of  $\mathbb{S}_t$ , as  $t \rightarrow T$  (see for instance the following figures), one will need the extension of Theorem 11.1 mentioned in Remark 11.20, in order to restart the flow.

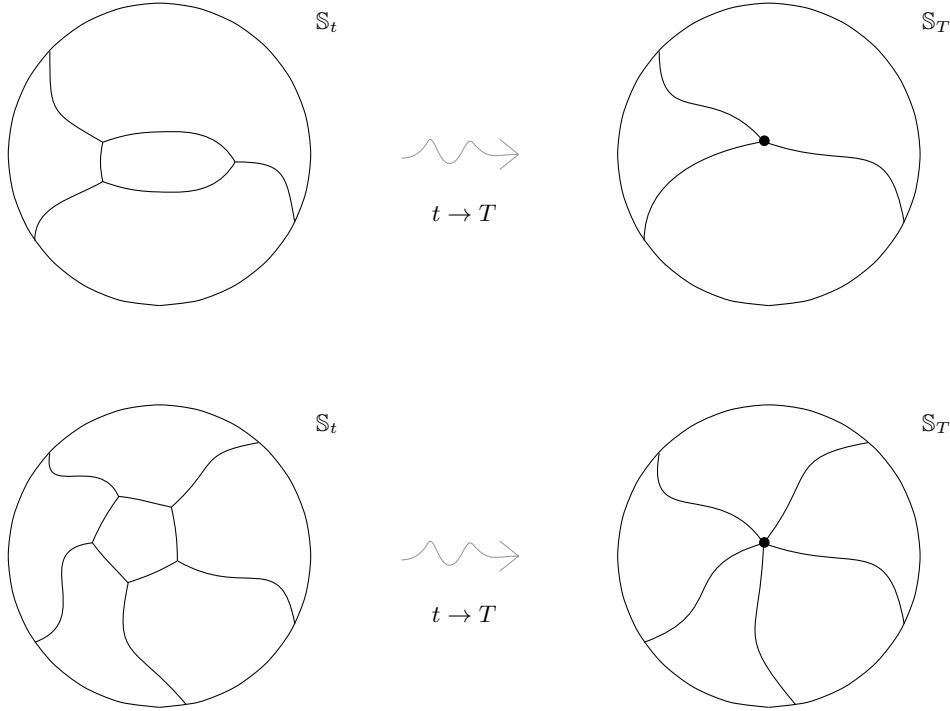


Figure 25: Less “nice” examples of collapse and convergence to non-regular networks  $\mathbb{S}_T$ .

We conclude this section by discussing the (conjectural) “generic” situation of singularity formation, in the sense that it should happen for a dense set of initial networks.

By numerical evidence (computing the lowest relevant eigenvalue of the Jacobi-field operator of the candidates – Dominic Descombes and Tom Ilmanen, *personal communication*) the *dynamically stable* shrinkers (meaning that “perturbing” the flow, the blow-up limit network remains the same) should

be only the line, the unit circle, the standard triod, the standard cross, the Brakke spoon, the lens and the “three-ray star” (see the figure below). Moreover, it can be actually rigorously proved that among the tree-like shrinkers, only the line, standard triod and standard cross are dynamically stable, any degenerate regular shrinker made of more than four halflines for the origin is dynamically unstable.

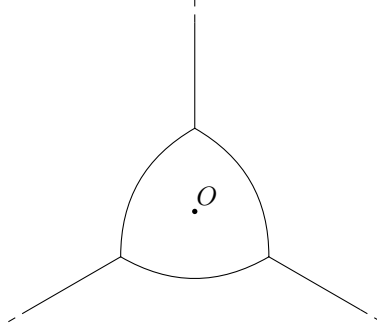


Figure 26: A “three-ray star” regular shrinker.

**Conjecture 10.39.** The “generic” singularities of the curvature flow of a network are (locally) asymptotically described by one of the above shrinkers.

We remark that if rescaling around a singular point  $x_0$ , we get one of the above shrinkers, the limit network  $\mathbb{S}_T$  is locally quite “nice”, since if it is a line or a standard triod, there is no singularity; if it is a circle, it means that the flow ends at the singularity; if it is a Brakke spoon, locally the flow produces a curve with an end-point in  $\Omega$  (see Figures 22 and 23), which we can reasonably “assume” it disappears at subsequent times and we have to deal with an empty network or with a curve containing an angle (as in Figure 19) that has a “natural” unique evolution, immediately smooth; if it is a standard cross, we can deal with the “new” 4-point by means of Theorem 11.1; if it is a lens,  $\mathbb{S}_T$  will be (locally) given by two  $C^1$  curves (smooth outside  $x_0$ ) concurring at the singular point without forming an angle (even if their curvature could be unbounded, getting to  $x_0$ , if Problem 10.34 has a negative answer); finally, if the shrinker is a three-ray star, the limit network  $\mathbb{S}_T$  is locally a triod at  $x_0$  with angles of 120 degrees, by Proposition 10.30 (also in this case the curvature could be unbounded getting close to  $x_0$ ). Notice that in these last two cases, even if apparently “nice”, we have anyway to use Theorem 11.1 in order to restart the flow, since the curves are not necessarily  $C^2$ .

However, we remark that in all these cases (and in particular in the most “delicate” ones: cross, lens and three-ray star, when we need to apply Theorem 11.1, or its extension mentioned in Remark 11.20) the associated limit network  $\mathbb{S}_T$  (if not empty and “cutting” away a curve if it ends in a 1-point in  $\Omega$ ) has either a 4-point with angles of 120/60 degrees, or a 120 degrees 3-point, or a 2-point with no angle. In particular, the cone generated by inner unit tangent vectors of the concurring curves at such point form, respectively, either a standard cross, or a regular triod or a line. Since, as we will see in the next section, the curvature flow produced by Theorem 11.1 is associated to a regular self-similarly expanding network (see Definition 11.11) originating from such cone, which in these special cases it is unique (see the end of Section 11.2 and Problems 11.7, 11.18), it is natural to expect that also the flow produced by such theorem is unique, which would give a unique “canonical” way to continue the flow in the (conjectural) generic situation.

## 11 Short time existence III – Non-regular networks

Recall that we said in Section 2 that a  $C^1$  network is *non-regular* if at the multiple points more than three curves are allowed to meet or, even if only three curves meet at a point, the 120 degrees condition is not satisfied.

The problem of defining and finding a curvature flow (as smooth as possible), starting from an initial non-regular network arises naturally also in connection with the problem of “restarting” the flow after

a singularity, discussed in the previous section. This clearly requires a definition of solution slightly different from Definitions 2.10 and 2.11 in a time interval  $[0, T)$  (as we will see, provided by Brakke flows, Definition 6.12), even asking that Definition 2.6 still holds for every positive time. Recently, T. Ilmanen, A. Neves and the last author have shown that such an evolution exists (see [47, Theorem 1.1]) for the special family of non-regular networks such that at every multiple point the exterior unit tangent vectors are mutually distinct. Notice that this is not restrictive for the “restarting” problem, taking into account the conclusions of Theorems 10.37 and 10.38.

**Theorem 11.1.** *Let  $\mathbb{S}_0$  be a possibly non-regular, embedded,  $C^1$  network with bounded curvature, which is  $C^2$  away from its multi-points and such that the exterior unit tangent vectors of the concurring curves at every multi-point are mutually distinct. Then, there exist  $T > 0$  and a smooth curvature flow of connected regular networks  $\mathbb{S}_t$ , locally tree-like, for  $t \in (0, T)$ , such that  $\mathbb{S}_t$  for  $t \in [0, T)$  is a regular Brakke flow. Moreover, away from the multi-points of  $\mathbb{S}_0$  the convergence of  $\mathbb{S}_t$  to  $\mathbb{S}_0$ , as  $t \rightarrow 0$ , is in  $C_{\text{loc}}^2$  (or as smooth as  $\mathbb{S}_0$ ). Furthermore, there exists a constant  $C > 0$  such that  $\sup_{\mathbb{S}_t} |k| \leq C/\sqrt{t}$  and the length of the shortest curve of  $\mathbb{S}_t$  is bounded from below by  $C\sqrt{t}$ .*

**Remark 11.2.** To be more precise, letting  $\tau(x, t)$  be a unit tangent vector at  $x \in \mathbb{S}_t$ , we define the sets  $G_t$  as follows,

$$G_t = \{(x, \tau(x, t)) \mid x \in \mathbb{S}_t\} \cup \{(x, -\tau(x, t)) \mid x \in \mathbb{S}_t\} \subset \mathbb{R}^2 \times \mathbb{S}^1,$$

for every  $t \in [0, T)$ . The convergence of  $\mathbb{S}_t \rightarrow \mathbb{S}_0$  in the previous theorem is actually in the sense of *varifolds*, that is, the Hausdorff measures  $\mathcal{H}^1 \llcorner G_t$  converge to  $\mathcal{H}^1 \llcorner G_0$ , when  $t \rightarrow 0$ , as measures on  $\mathbb{R}^2 \times \mathbb{S}^1$  (see [77], for the general definition). It is easy to see that this implies that  $\mathcal{H}^1 \llcorner \mathbb{S}_t \rightarrow \mathcal{H}^1 \llcorner \mathbb{S}_0$ , as  $t \rightarrow 0$ , as measures on  $\mathbb{R}^2$ , hence, there is no instantaneous loss of mass of the network at the starting time.

Clearly, around a non-regular multi-point the  $C^1$ -convergence is not possible, since for every  $t > 0$ , the networks  $\mathbb{S}_t$  are regular, so they satisfy the 120 degrees condition and that would pass to the limit. Varifold-convergence is anyway a sort of “weak”  $C^1$ -convergence, slightly stronger than simply asking that  $\mathcal{H}^1 \llcorner \mathbb{S}_t \rightarrow \mathcal{H}^1 \llcorner \mathbb{S}_0$ , as  $t \rightarrow 0$ .

In this section we will present an outline of the proof of this result. It depends crucially on an expander monotonicity formula which implies that self-similarly expanding flows are “dynamically stable”. The monotone integral quantity we will consider has been applied previously by A. Neves in the setting of Lagrangian mean curvature flow, see [66–68]. Other main ingredients are the local regularity theorem 9.3 and the pseudolocality Theorem 10.31 (see [47, Theorem 1.5]). We underline that for curves moving in the plane, this latter can be replaced by S. Angenent’s *intersection counting theorem*, see [8, Proposition 1.2], [7, Section 2] and [5] for the proof.

By the assumptions, at any multi-point of an initial network  $\mathbb{S}_0$ , the cone generated (at such point) by the interior unit normal vectors of the concurring curves consists of a finite number of distinct halflines. The natural evolution of such cone is a self-similarly expanding curvature flow, due to the scaling invariance of this particular initial network. It was actually shown by O. Schnürer and the last author, see [75] and R. Mazzeo and M. Saez [64], that such a regular, tree-like, self-similarly expanding solution always exists.

The strategy is then as follows: we “glue in”, around each possibly non-regular multi-point of the initial network  $\mathbb{S}_0$ , a (piece of a) smooth, self-similarly expanding, tree-like, connected regular network at the scale  $\sqrt{\xi}$  (in a ball of radius proportional to  $\sqrt{\xi}$ ), corresponding to the cone generated by the interior unit tangent vectors of the concurring curves of  $\mathbb{S}_0$  at the multi-point, to obtain an approximating  $C^2$  regular network  $\mathbb{S}_0^\xi$  (satisfying the compatibility conditions of every order, see Definition 4.15). The curvature of  $\mathbb{S}_0^\xi$  is thus of order  $1/\sqrt{\xi}$  and the shortest curve has length proportional to  $\sqrt{\xi}$ . Then, the standard short time existence result yields a smooth curvature flow  $\mathbb{S}_t^\xi$  up to a positive time  $T_\xi$ .

To prove then that these approximating flows actually exist for a time  $T > 0$ , independent of  $\xi$ , we make use of the expander monotonicity formula to show that the flows  $\mathbb{S}_t^\xi$  stay close to the corresponding self-similarly expanding flows, in an integral sense, around each multi-point. This gives that the curvature is bounded by  $C/\sqrt{t}$  up to a fixed time  $T > 0$ , together with a lower bound on the length of the shortest curve. Thus, we can pass to the limit, as  $\xi \rightarrow 0$ , to obtain the desired curvature flow.

**Remark 11.3.** The Brakke flow provided by the above theorem is not necessarily *with equality* (see Definition 6.12). Indeed, for instance, if  $\mathbb{S}_0$  is a standard cross (see Figure 9) and  $\varphi$  a test function such that

$0 \leq \varphi \leq 1$ ,  $\varphi = 1$  on  $B_1(0)$  and  $\varphi = 0$  outside of  $B_2(0)$ , letting  $\mathbb{S}_t = \sqrt{2t}\mathbb{S}_0$  be the regular expander “exiting” from  $\mathbb{S}_0$  (which is the curvature flow given by the theorem), if  $\mathbb{S}_t$  would be a regular Brakke flow with equality, it should be

$$\frac{d}{dt} \int_{\mathbb{S}_t} \varphi ds \Big|_{t=0} = - \int_{\mathbb{S}_0} \varphi k^2 ds + \int_{\mathbb{S}_0} \langle \nabla \varphi | \underline{k} \rangle ds = 0,$$

by equation (6.2) and since  $\mathbb{S}_0$  has no curvature.

Anyway, by the mean value theorem, for any  $t > 0$  there holds

$$\frac{\int_{\mathbb{S}_t} \varphi ds - \int_{\mathbb{S}_0} \varphi ds}{t} = - \int_{\mathbb{S}_\theta} \varphi k^2 ds + \int_{\mathbb{S}_\theta} \langle \nabla \varphi, \underline{k} \rangle ds,$$

for some  $0 < \theta < t$ . It is then easy to see, by the self-similarity property of  $\mathbb{S}_t = \sqrt{2t}\mathbb{S}_0$ , that the first term on the right hand side of this formula goes to  $-\infty$  and the second one stays bounded, hence,

$$\frac{d}{dt} \int_{\mathbb{S}_t} \varphi ds \Big|_{t=0} = \limsup_{t \rightarrow 0} \frac{\int_{\mathbb{S}_t} \varphi ds - \int_{\mathbb{S}_0} \varphi ds}{t} = -\infty,$$

which is a contradiction.

*Remark 11.4.* In writing this paper, we got informed that the hypothesis on the non-coincidence of two or more exterior unit tangent vectors can actually be removed (Tom Ilmanen, *personal communication*).

*Remark 11.5.* The a priori “choice” of gluing in only connected regular self-similarly expanding networks, hence obtaining a connected network flows, has a clear “physical” meaning: it ensures that initially separated regions remain separated during the flow. Instead, the a priori choice of using only tree-like self-similarly expanding networks excludes the formation of new bounded regions in the flow. Indeed, from a 7-point one could try (this is only conjectural, actually, the line of Theorem 11.1 does not work in this case) to get a flow with a new heptagonal region, by gluing in a symmetric self-similarly expanding network with an heptagonal region, following the construction of Theorem 11.1 described above.

Anyway, it can be seen that all the connected, regular self-similarly expanding networks containing a bounded region must have at least seven unbounded halflines. This because, by means of the same arguments of Section 8.2 (Remark 8.13), every bounded region of a regular self-similarly expanding network is bounded by at least seven curves. This clearly implies that from a multi-point of order less than six, the flow produced by Theorem 11.1 is always locally tree-like, even if the line of proof (and at the moment it is not) could be adapted to glue in *any* self-similarly expanding network (that is, possibly also a non tree-like one, in general). It is then a natural question if actually a multi-point with more than five (or possibly more than six) concurring curve can appear in the limit network  $\mathbb{S}_T$ , as  $t \rightarrow T$ , described in Theorem 10.37 of the previous section. This is related to finding a regular (possibly degenerate) shrinker with more than five (or maybe six) unbounded halflines.

**Open Problem 11.6.** Do there exist (possibly degenerate) regular shrinkers with more that five (or six) unbounded halflines?

In the situation such that there is a single expander coming out from the cone generated by the inner unit tangent vectors of the concurring curves to a multi-point, it is natural to conjecture that Theorem 11.1 produces a unique evolution.

**Open Problem 11.7.** If there is a unique regular expander asymptotic to the family of halflines generated by the inner unit tangent vectors of the concurring curves to a multi-point of  $\mathbb{S}_0$ , then Theorem 11.1 produces a *unique* curvature flow?

## 11.1 Expander monotonicity formula

Let  $\mathbb{S}_t$  be a curvature flow of tree-like regular networks. The tangent vector of  $\mathbb{S}_t$  makes with the  $x$ -axis an angle  $\bar{\theta}_t$  which, away from the triple junctions, is a well defined function up to a multiple of  $\pi$ , since we do not care about orientation. Because at the triple junctions the angle jumps by  $2\pi/3$ , there is a well

defined function  $\theta_t$  which is continuous on  $\mathbb{S}_t$  and coincides with  $\bar{\theta}_t$  up to a multiple of  $\pi/3$ . We identify the plane  $\mathbb{R}^2$  with  $\mathbb{C}$ , thus

$$\underline{k} = J\tau \partial_s \theta_t = \nu \partial_s \theta_t ,$$

where  $J$  is the complex structure.

Let  $\mathcal{L} = xdy - ydx$  be the Liouville form on  $\mathbb{R}^2$ . Since we assumed that  $\mathbb{S}_t$  has no loops, we can find a function  $\beta_t$ , unique up to a time-dependent constant, such that

$$d\beta_t = \mathcal{L}|_{\mathbb{S}_t} .$$

We can modify the time-dependent constant in order that the following evolution equations hold, see [47, Lemma 3.1].

**Lemma 11.8.** *The following evolution equations hold away from the triple junctions:*

$$\frac{d\theta_t}{dt} = \partial_{ss}^2 \theta_t + \partial_s \theta_t \langle \tau | X \rangle ,$$

$$\frac{d\beta_t}{dt} = \partial_{ss}^2 \beta_t + \partial_s \beta_t \langle \tau | X \rangle - 2\theta_t ,$$

where  $X = \underline{k} + \lambda\tau$  is the velocity of the evolution.

Note that this implies that the function  $\alpha_t = \beta_t + 2t\theta_t$  satisfies the evolution equation

$$\frac{d\alpha_t}{dt} = \partial_{ss}^2 \alpha_t + \partial_s \alpha_t \langle \tau | X \rangle .$$

Furthermore,  $J\tau \partial_s \alpha_t = \nu \partial_s \alpha_t = -x^\perp + 2t\underline{k}$ , which exactly vanishes on a self-similarly expanding network. With a computation similar to the one leading to Huisken's monotonicity formula (7.1), we arrive at the following result, see [47, Lemma 3.2].

**Lemma 11.9** (Expander monotonicity formula). *The following identity holds*

$$\frac{d}{dt} \int_{\mathbb{S}_t} \alpha_t^2 \rho_{x_0, t_0}(x, t) ds = - \int_{\mathbb{S}_t} 2 |x^\perp - 2t\underline{k}|^2 \rho_{x_0, t_0}(x, t) ds - \int_{\mathbb{S}_t} \alpha_t^2 \left| \underline{k} + \frac{(x - x_0)^\perp}{2(t_0 - t)} \right|^2 \rho_{x_0, t_0}(x, t) ds ,$$

for some constant  $C$ .

In the later applications, the evolving networks will be only locally tree-like, that is, only locally without loops. In order to apply the above monotonicity formula, it will need to be localized. We assume that  $\mathbb{S}_t \cap B_4(x_0)$  does not contain any closed loop for all  $0 \leq t < T$ . We define  $\beta_t$  locally on  $\mathbb{S}_t \cap B_4(x_0)$  and we let  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a smooth cut-off function such that  $\varphi = 1$  on  $B_2(x_0)$ ,  $\varphi = 0$  on  $\mathbb{R}^2 \setminus B_3(x_0)$  and  $0 \leq \varphi \leq 1$ . Then, we have the following localized version of Lemma 11.9, see [47, Lemma 3.3].

**Lemma 11.10** (Localized expander monotonicity formula). *The following estimate holds,*

$$\frac{d}{dt} \int_{\mathbb{S}_t} \varphi \alpha_t^2 \rho_{x_0, t_0}(x, t) ds \leq - \int_{\mathbb{S}_t} \varphi |x^\perp - 2t\underline{k}|^2 \rho_{x_0, t_0}(x, t) ds + C \int_{\mathbb{S}_t \cap (B_3(x_0) \setminus B_2(x_0))} \alpha_t^2 \rho_{x_0, t_0}(x, t) ds .$$

## 11.2 Self-similarly expanding networks

**Definition 11.11.** A regular  $C^2$  open network  $\mathbb{E}$  is called a *regular expander* if at every point  $x \in \mathbb{E}$  there holds

$$\underline{k} = x^\perp . \tag{11.1}$$

This relation is called the *expanders equation*.



The name comes from the fact that if  $\mathbb{E}$  is a regular expander, then  $\mathbb{E}_t = \sqrt{2t} \mathbb{E}$  describes a *self-similarly expanding* curvature flow of regular networks in  $(0, +\infty)$ , with  $\mathbb{E} = \mathbb{E}_{1/2}$ . Viceversa, if  $\mathbb{E}_t$  is a self-similarly expanding curvature flow of regular networks in the time interval  $(0, +\infty)$ , then  $\mathbb{E}_{1/2}$  is a regular expander, that is,  $\mathbb{E}_{1/2}$  satisfies equation (11.1).

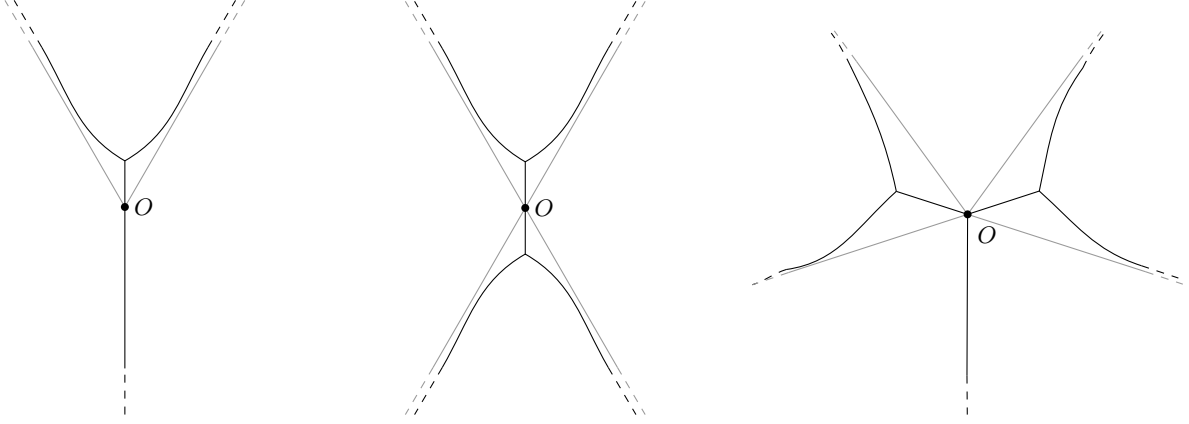


Figure 27: Examples of tree-like regular expanders with 3, 4, 5 asymptotic halflines (in gray).

**Lemma 11.12.** *A regular expander is a critical point of the length functional with respect to the negatively curved metric*

$$g = e^{|x|^2} (dx_1^2 + dx_2^2).$$

*Proof.* See [64, Proposition 2.3] or [47, Lemma 4.1].  $\square$

By studying the ODE satisfied along each curve, one can easily show that an expander cannot be compact, all its curves are smooth and each noncompact curve must be asymptotic to a halfline. Moreover, it is trivial that the family of the asymptotic halflines of the open networks of a self-similarly expanding curvature flow  $\mathbb{E}_t$  is the same for all  $t \in (0, +\infty)$  and, by a direct maximum principle argument, one can prove exponential decay. We summarize all this in the following lemma, for a proof see [47, Lemma 4.3].

**Lemma 11.13.** *Let  $P$  be a finite union of distinct halflines meeting at the origin and  $\mathbb{E}$  a regular expander, such that each noncompact curve of  $\mathbb{E}$  is asymptotic in Hausdorff distance to one of the halflines of  $P$ . Then, there exists an  $r_0 > 0$  large enough such that each noncompact curve  $\sigma$  of  $\mathbb{E}$  corresponds to a connected component of  $\mathbb{E} \setminus B_{r_0}(0)$  and can be parametrized as*

$$\sigma(\ell) = \ell e^{i\omega} + u(\ell) e^{i(\omega + \pi/2)} \quad \text{for } \ell \geq r_0.$$

where  $\{\ell e^{i\omega} \mid \ell \geq 0\}$  is a halfline of  $P$  and  $\lim_{\ell \rightarrow +\infty} u(\ell) = 0$ . Moreover, the decay of  $u$  is given by

$$|u(\ell)| \leq C_0 e^{-\ell^2/2}, \quad |u'(\ell)| \leq C_1 \ell^{-1} e^{-\ell^2/2}, \quad |u''(\ell)| \leq C_2 e^{-\ell^2/2}$$

and

$$|u'''(\ell)| \leq C_3 \ell e^{-\ell^2/2}, \quad |u''''(\ell)| \leq C_4 \ell^2 e^{-\ell^2/2},$$

where each  $C_i$  depends only on  $r_0$ ,  $u(r_0)$  and  $u'(r_0)$ .

Then, it is easy to see that for every smooth self-similarly expanding curvature flow  $\mathbb{E}_t$ , letting  $P$  be the network given by the finite union of the distinct (common) asymptotic halflines of  $\mathbb{E}_t$ , meeting at the origin, we have  $\mathbb{E}_t \rightarrow P$ , as  $t \rightarrow 0$ , in  $C_{\text{loc}}^\infty(\mathbb{R}^2 \setminus \{0\})$ . We say that  $P$  is the *generator* of the flow  $\mathbb{E}_t$  or that  $\mathbb{E}_t$  is a (possibly not unique) curvature flow of  $P$  in the time interval  $[0, +\infty)$ .

Conversely, if we consider a network  $P$  given by a finite number of distinct halflines meeting at the origin and we assume that we have a smooth curvature flow  $\mathbb{S}_t$ , for  $t \in (0, T)$  such that  $\mathbb{S}_t \rightarrow P$ , as  $t \rightarrow 0$ , in  $C_{\text{loc}}^\infty(\mathbb{R}^2 \setminus \{0\})$ , then, the parabolically rescaled flows

$$\mathbb{S}_t^\mu = \mu \mathbb{S}_{\mu^{-2}t}$$

also satisfy  $\mathbb{S}_t^\mu \rightarrow P$ , as  $t \rightarrow 0$ , for any  $\mu > 0$ , since  $P$  is invariant under rescalings. Thus, supposing that the flow  $\mathbb{S}_t$  is unique in some “appropriate class” with initial condition  $P$ , we obtain that  $T = +\infty$  and  $\mathbb{S}_t = \mathbb{S}_t^\mu$ , for any  $\mu, t > 0$ . This is like to say that  $\mathbb{S}_t = \sqrt{2t} \mathbb{S}_{1/2}$ , that is,  $\mathbb{S}_t$  is a self-similarly expanding curvature flow of regular networks, for  $t \in (0, +\infty)$  and  $P$  is its generator. As we said, the family of distinct (common) asymptotic halflines of all  $\mathbb{S}_t$  coincides with the family of halflines of  $P$ .

*Remark 11.14.* Notice that the generator of a self-similarly expanding curvature flow of networks is uniquely defined, while, for a network  $P$  composed by a finite number of halflines for the origin, there could be several self-similarly expanding curvature flows of regular networks having  $P$  as generator (see Figure 28).

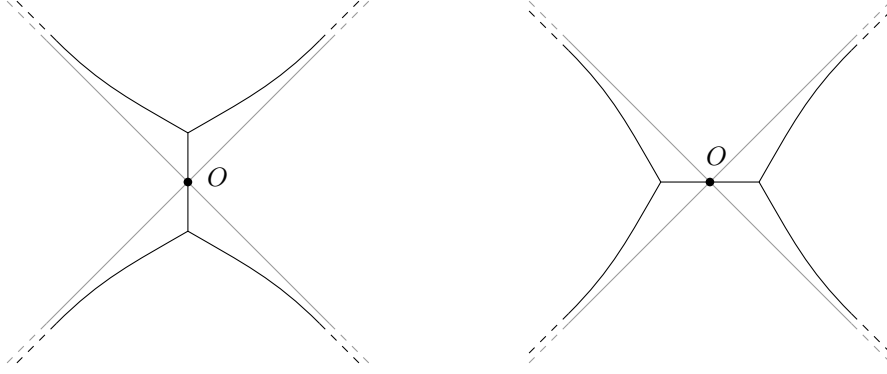


Figure 28: An example of two different tree-like regular expanders (not in the same “topological class” – see below) with the same asymptotic halflines (in gray).

Given  $P = \bigcup_{j=1}^n P_j$  where  $P_j$  are halflines for the origin, in [75] it was shown that for  $n = 3$  there exists a *unique* tree-like, regular expander  $\mathbb{E}$  asymptotic to  $P$  (if  $P$  is a standard triod such an expander  $\mathbb{E}$  is  $P$  itself), in the case  $n > 3$  the existence of tree-like, connected, regular expanders was shown by Mazzeo-Saez [64].

A key fact is that two regular expanders with the same “topological structure” and which are asymptotic to the same family of halflines, have to be identical. More precisely:

**Definition 11.15.** We say that two regular expanders  $\mathbb{E}_0$  and  $\mathbb{E}_1$  are *asymptotic to each other* if their ends are asymptotic to the same halflines.

We say that two regular expanders  $\mathbb{E}_0$  and  $\mathbb{E}_1$  are in the *same topological class* if there is a smooth family of maps

$$F_\nu : \mathbb{E}_0 \rightarrow \mathbb{R}^2, \quad 0 \leq \nu \leq 1$$

such that  $F_0$  is the identity,  $F_1(\mathbb{E}_0) = \mathbb{E}_1$ , the distance between any two triple junctions of  $F_\nu(\mathbb{E}_0)$  is uniformly bounded below and

$$\lim_{r_0 \rightarrow +\infty} \sup \{ |\partial F_\nu(x) / \partial \nu| \mid x \in \mathbb{E}_0 \setminus B_{r_0}(0) \} = 0 \text{ for every } 0 \leq \nu \leq 1.$$

Notice that two regular expanders in the same topological class are asymptotic to each other.

**Theorem 11.16.** *If  $\mathbb{E}_0$  and  $\mathbb{E}_1$  are two regular expanders in the same topological class, then they coincide.*

*Proof.* We work in the negatively curved metric in the plane

$$g = e^{|x|^2} (dx_1^2 + dx_2^2),$$

such that each curve of a regular expander is a geodesic in this metric.

Let  $\{x_i^0\}$  and  $\{x_i^1\}$  denote the triple junctions (a finite set) of  $\mathbb{E}_0$  and  $\mathbb{E}_1$ , respectively. As the networks are in the same topological class, we can rearrange the elements of  $\{x_i^0\}$  so that each  $x_i^0$  is connected to  $x_i^1$  by the existing deformation  $F_\nu$  of  $\mathbb{E}_0$  into  $\mathbb{E}_1$ . Denote by  $x_i^\xi$ , for  $\xi \in [0, 1]$ , the unique geodesic connecting these points.

For each  $\xi$  we consider the network  $\mathbb{E}_\xi$  such that if  $x_i^0$  is connected to  $x_k^0$  by a geodesic, then  $x_i^\xi$  is connected to  $x_k^\xi$  through a geodesic as well. To handle the noncompact curves we proceed as follows. Let  $P_j$  denote a common asymptotic halfline to  $\mathbb{E}_0$  and  $\mathbb{E}_1$ , which means that there are geodesics  $\psi_0 \subset \mathbb{E}_0$ ,  $\psi_1 \subset \mathbb{E}_1$  asymptotic to  $P_j$  at infinity and starting at some points  $x_i^0$  and  $x_i^1$  respectively. Define then, for every  $\xi \in (0, 1)$ , the curve  $\psi_\xi \subset \mathbb{E}_\xi$  to be the unique geodesic starting at  $x_i^\xi$  and asymptotic to  $P_j$ . This gives a deformation of the curve  $\psi_0$  to  $\psi_1$ .

Hence, we have constructed a smooth family of networks with only triple junctions  $\mathbb{E}_\xi$ , for  $\xi \in [0, 1]$ , “connecting”  $\mathbb{E}_0$  and  $\mathbb{E}_1$  and such that:

1. The triple junctions  $\{x_i^\xi\}$  of  $\mathbb{E}_\xi$  connect the triple junctions of  $\mathbb{E}_0$  to the ones of  $\mathbb{E}_1$  and, for each index  $i$  fixed, the path  $x_i^\xi$ , with  $\xi \in [0, 1]$ , is a geodesic with respect to the metric  $g$ .
2. Each curve of  $\mathbb{E}_\xi$  is a geodesic of  $(\mathbb{R}^2, g)$ .
3. There is  $r_0 > 0$  large enough so that  $\mathbb{E}_\xi \setminus B_{r_0}(0)$  has  $n$  connected components, each asymptotic to a halfline  $P_j$ , for  $j = 1, 2, \dots, n$ . We can find angles  $\omega_j$  such that each end of  $\mathbb{E}_\xi$  becomes parametrized as

$$\mathbb{E}_\xi(\ell) = \ell e^{i\omega_j} + u_{j,\xi}(\ell) e^{i(\omega_j + \pi/2)} \quad \text{for } \ell \geq r_0.$$

This follows from Lemma 11.13.

4. The vector field along  $\mathbb{E}_\xi$

$$X_\xi(\ell) = \frac{\partial}{\partial \xi} \mathbb{E}_\xi(\ell)$$

is continuous, smooth when restricted to each curve and

$$|X_\xi(\ell)| = O(e^{-\ell^2/2}), \quad |\nabla X_\xi(\ell)| = O(\ell^{-1} e^{-\ell^2/2}),$$

uniformly in  $\xi \in [0, 1]$ , where the gradient is computed along  $\mathbb{E}_\xi$  with respect to the metric  $g$ . Moreover,

$$\alpha_{j,\xi}(\ell) = \frac{\partial u_{j,\xi}(\ell)}{\partial \xi}$$

satisfies

$$|\alpha_{j,\xi}(\ell)| = O(e^{-\ell^2/2}) \quad |\alpha'_{j,\xi}(\ell)| = O(\ell^{-1} e^{-\ell^2/2}).$$

It is enough to provide justification for the second set of estimates. For ease of notation we omit the indices  $j$  and  $\xi$  on  $\alpha_{j,\xi}$  and  $u_{j,\xi}$ . By linearizing the equation for an expanding graph, see [75, Equation (2.3)], we have

$$\alpha'' = (1 + [u']^2)(\alpha - \ell \alpha') + 2u' \alpha' (u - \ell u').$$

We can assume without loss of generality that  $\alpha(r_0) \geq 0$ . Moreover, it follows from our construction that

$$\lim_{\ell \rightarrow +\infty} |\alpha(\ell)| + |\alpha'(\ell)| = 0.$$

A simple application of the maximum principle shows that  $\alpha$  can not have negative local minimum or a positive local maximum. Hence,  $\alpha \geq 0$  and  $\alpha' \leq 0$ . We can assume that  $u' \leq 0$  (see the proof of Lemma 11.13). The function  $\beta = \alpha - \ell \alpha'$  thus satisfies

$$\beta' = -\ell(1 + [u']^2)\beta - 2\ell u' \alpha' \leq -x\beta$$

and integration of this inequality gives the conclusion.

Denote by  $L$  the length functional with respect to the metric  $g$  and consider the family of functions

$$W_r(\xi) = L(\mathbb{E}_\xi \cap B_{2r_0}(0)) + \sum_{j=1}^n \int_{2r_0}^r e^{[\ell^2 + u_{j,\xi}^2(\ell)]/2} \sqrt{1 + [u'_{j,\xi}(\ell)]^2} d\ell - n \int_{2r_0}^r e^{\ell^2/2} d\ell.$$

The decays given in Lemma 11.13 imply the existence of a constant  $C$  such that for every  $r \leq \bar{r}$

$$\|W_r - W_{\bar{r}}\|_{C^3} \leq Ce^{-r}, \quad (11.2)$$

so, when  $r \rightarrow +\infty$ , the sequence of functions  $W_r : [0, 1] \rightarrow \mathbb{R}$  converges uniformly in  $C^2$  to a function  $W : [0, 1] \rightarrow \mathbb{R}$ . Furthermore, if  $\xi = 0$  or  $\xi = 1$ , we have, combining Lemma 11.13 with property 4 above, that

$$\lim_{r \rightarrow +\infty} \frac{dW_r(\xi)}{d\xi} = 0,$$

thus,  $W$  has a critical point when  $\xi = 0$  or  $\xi = 1$ .

A standard computation shows that on each compact curve of  $\mathbb{E}_\xi$  we have (after reparametrization proportional to arclength)

$$\begin{aligned} \frac{d^2}{d\xi^2} \int_a^b \sqrt{g(\mathbb{E}'_\xi, \mathbb{E}'_\xi)} dl &= \int_a^b |\mathbb{E}'_\xi|^{-1} (|(\nabla_{\mathbb{E}'_\xi} X_\xi)^\perp|^2 - \text{Rm}(X_\xi, \mathbb{E}'_\xi, \mathbb{E}'_\xi, X_\xi)) dl + |\mathbb{E}'_\xi|^{-1} g(\nabla_{X_\xi} X_\xi, \mathbb{E}'_\xi) \Big|_a^b \\ &= \int_a^b |\mathbb{E}'_\xi|^{-1} (|(\nabla_{\mathbb{E}'_\xi} X_\xi)^\perp|^2 - \text{Rm}(X_\xi, \mathbb{E}'_\xi, \mathbb{E}'_\xi, X_\xi)) dl, \end{aligned}$$

where  $\mathbb{E}'_\xi = d\mathbb{E}_\xi/dl$ , we used property 1 above and all the geometric quantities are computed with respect to the metric  $g$  ( $\text{Rm}$  is the Riemann tensor of  $(\mathbb{R}^2, g)$ ). Combining this identity with property 4, we have

$$\frac{d^2 W_r(\xi)}{d\xi^2} = \int_{\mathbb{E}_\xi \cap B_r(0)} |\mathbb{E}'_\xi|^{-2} (|(\nabla_{\mathbb{E}'_\xi} X_\xi)^\perp|^2 - \text{Rm}(X_\xi, \mathbb{E}'_\xi, \mathbb{E}'_\xi, X_\xi)) dl + O(e^{-r}).$$

As  $(\mathbb{R}^2, g)$  is negatively curved, more precisely, its Gaussian curvature is equal to  $-e^{-|x|^2}$ , the integrals above are bounded independently of  $r > 2r_0$ . Therefore, by means of estimate (11.2), we obtain

$$\frac{d^2 W(\xi)}{d\xi^2} = \int_{\mathbb{E}_\xi} |\mathbb{E}'_\xi|^{-2} (|(\nabla_{\mathbb{E}'_\xi} X_\xi)^\perp|^2 - \text{Rm}(X_\xi, \mathbb{E}'_\xi, \mathbb{E}'_\xi, X_\xi)) dl \geq 0,$$

where the last inequality comes from the fact that  $(\mathbb{R}^2, g)$  is negatively curved. It follows that  $W : [0, 1] \rightarrow \mathbb{R}$  is a convex function with two critical points at  $\xi = 0$  and  $\xi = 1$ , hence, it is identically constant. The last formula above then implies that the vector field  $X_\xi$  must be a constant multiple of  $\mathbb{E}'_\xi$ , hence, it must vanish at all triple junctions. The fact that  $X_\xi$  is continuous implies that  $X_\xi$  is identically zero and this proves that all the networks  $\mathbb{E}_\xi$  coincide, for  $\xi \in [0, 1]$ , in particular  $\mathbb{E}_0 = \mathbb{E}_1$ , which is the desired result.  $\square$

**Corollary 11.17.** *If  $P = \bigcup_{j=1}^4 P_j$  is a standard cross, then there exists a unique, connected, tree-like, regular expander asymptotic to  $P$ .*

*Proof.* In this case it is easy to see that there are only two possible topological classes of connected regular expanders asymptotic to  $P$  (analogous to the two situations depicted in Figure 28), but since every unbounded curve cannot change its convexity (as for the shrinkers, by analyzing the expanders equation (11.1)), if two such curves are contained in the angle of 120 degrees of the standard cross, when they concur at a 3-point they must form an angle larger the 120 degrees, which is a contradiction, hence such topological class is forbidden.

Thus, only one topological class is allowed and it contains only one regular expander (with two symmetry axes), by Theorem 11.16.  $\square$

Then, we have naturally the following special case of Problem 11.7.

**Open Problem 11.18.** *If the inner unit tangent vectors of the concurring curves to a 4-point of  $\mathbb{S}_0$  generate a standard cross, Theorem 11.1 produces a unique curvature flow?*

### 11.3 The proof of Theorem 11.1

Now let  $\mathbb{S}_0$  be a non-regular initial network with bounded curvature. For simplicity, let us assume that  $\mathbb{S}$  has only one non-regular multi-point at the origin.

If the multi-point consists of only two curves meeting at an angle different from  $\pi$  (remember that a zero angle is not allowed), then, by the work of Angenent [6–8], there exists a curvature flow starting at  $\mathbb{S}_0$ , satisfying the statement of Theorem 11.1: actually the angle is immediately smoothed and the two curves become a single smooth one.

So we can assume that at the origin at least three curves meet and let  $\tau_j$ , for  $j = 1, 2, \dots, n$ , be the exterior unit tangent vectors. We denote with

$$P_j = \{-\ell\tau_j \mid \ell \geq 0\}$$

the corresponding halflines and  $P = \bigcup_{j=1}^n P_j$ . Since  $\mathbb{S}_0$  has bounded curvature, we can assume, by scaling  $\mathbb{S}_0$  if necessary, that  $\mathbb{S}_0 \cap B_5(0)$  consists of  $n$  curves  $\sigma_j$  corresponding to the tangents  $\tau_j$  and if  $\omega_j$  is the angle that  $P_j$  makes with the  $x$ -axis, there is a function  $u_j$  such that  $\sigma_j$  can be parametrized (with a small error at the boundary of the ball  $B_5(0)$ ) as

$$\sigma_j = \{\ell e^{i\omega_j} + u_j(\ell) e^{i(\omega_j + \pi/2)} \mid 0 \leq \ell \leq 5\}.$$

Notice that the assumption that  $\mathbb{S}_0$  has bounded curvature implies

$$|u_j(\ell)| \leq C\ell^2 \quad \text{and} \quad |u'_j(\ell)| \leq C\ell,$$

for some constant  $C$ .

In [75] it was shown that for  $n = 3$  there exists a unique tree-like regular expander  $\mathbb{E}$  asymptotic to  $P = \bigcup_{j=1}^n P_j$ . In the case  $n > 3$ , the existence of tree-like, connected, regular expanders was shown by Mazzeo-Saez [64].

We remind that, thanks to Lemma 11.13, there exists  $r_0 > 0$  such that outside the ball  $B_{r_0}(0)$  the  $n$  noncompact curves  $\gamma_j$  of the regular expander  $\mathbb{E}$  can be parametrized as

$$\gamma_j = \{\ell e^{i\omega_j} + v_j(\ell) e^{i(\omega_j + \pi/2)} \mid \ell \geq r_0\},$$

where the functions  $v_j$  have the following decay:

$$|v_j(\ell)| \leq C_0 e^{-\ell^2/2}, \quad |v'_j(\ell)| \leq C_1 \ell^{-1} e^{-\ell^2/2}, \quad |v''_j(\ell)| \leq C_2 e^{-\ell^2/2}.$$

Consider now the rescaled expander  $\mathbb{E}_\xi = \sqrt{2\xi} \mathbb{E}$ , call  $\sigma_{j,\xi}$  be the curve of  $\mathbb{E}_\xi$  asymptotic to  $P_j$ , for every  $j = 1, 2, \dots, n$ , then

$$\sigma_{j,\xi} = \{\ell e^{i\omega_j} + v_{j,\xi}(\ell) e^{i(\omega_j + \pi/2)} \mid \ell \geq r_0 \sqrt{2\xi}\},$$

and we have the estimates

$$|v_{j,\xi}(\ell)| \leq C \sqrt{2\xi} e^{-\ell^2/4\xi}, \quad |v'_{j,\xi}(\ell)| \leq C \ell^{-1} \sqrt{2\xi} e^{-\ell^2/4\xi}, \quad |v''_{j,\xi}(\ell)| \leq C e^{-\ell^2/4\xi} / \sqrt{2\xi}.$$

In particular, choosing  $\xi$  small enough, we have  $r_0 \sqrt{2\xi} < 4$  and this holds in the annulus  $A(r_0 \sqrt{2\xi}, 4) = B_4(0) \setminus B_{r_0 \sqrt{2\xi}}(0)$ .

We now aim to construct the network  $\mathbb{S}_0^\xi$  by gluing  $\mathbb{E}_\xi = \sqrt{2\xi} \mathbb{E}$  into  $\mathbb{S}_0$  (more precisely  $\mathbb{E}_\xi \cap B_{r_0 \sqrt{2\xi}}(0)$ , for  $\xi$  small enough). We define the network  $\mathbb{S}_0^\xi$  that coincides with  $\mathbb{E}_\xi$  in  $B_{r_0 \sqrt{2\xi}}(0)$  and with  $\mathbb{S}_0$  outside  $B_4(0)$ , while in the “gluing” annulus  $A(r_0 \sqrt{2\xi}, 4)$ , in a way we “interpolate” between the two networks. Precisely, letting  $\varphi : \mathbb{R}^+ \rightarrow [0, 1]$  be a cut-off function such that  $\varphi = 1$  on  $(0, 1]$  and  $\varphi = 0$  on  $[2, +\infty)$ , we define  $\mathbb{S}_0^\xi$  in  $A(r_0 \sqrt{2\xi}, 4)$  via the graph function  $u_{j,\xi}$  as follows, for  $\ell \in [r_0 \sqrt{2\xi}, 4]$ ,

$$u_{j,\xi}(\ell) = \varphi(\xi^{-1/4} \ell) v_{j,\xi}(\ell) + (1 - \varphi(\xi^{-1/4} \ell)) u_j(\ell).$$

That is,

$$\mathbb{S}_0^\xi \cap A(r_0 \sqrt{2\xi}, 4) = \{\ell e^{i\omega_j} + u_{j,\xi}(\ell) e^{i(\omega_j + \pi/2)} \mid r_0 \sqrt{2\xi} \leq \ell \leq 4\}$$

(with a small error at the borders of the annulus  $A(r_0\sqrt{2\xi}, 4)$ ).

By construction, every network  $\mathbb{S}_0^\xi$  has the same regularity of  $\mathbb{S}_0$ , it is regular and satisfies all the compatibility conditions of every order (see Definition 4.15), it is locally a tree and it can be checked easily that it satisfies the following properties, for every  $\xi$  smaller than some  $\xi_0 > 0$ :

$\mathcal{P}1$ . There is a constant  $D_1$ , independent of  $\xi$ , such that

$$\mathcal{H}^1(\mathbb{S}_0^\xi \cap B_r(x)) \leq D_1 r,$$

for all  $x \in \mathbb{R}^2$  and  $r > 0$ .

$\mathcal{P}2$ . There is a constant  $D_2$  independent of  $\xi$ , such that for every  $x \in \mathbb{S}_0^\xi$ ,

$$|\theta_0^\xi(x)| + |\beta_0^\xi(x)| \leq D_2(|x|^2 + 1),$$

where  $\theta_0^\xi$  and  $\beta_0^\xi$  are the “angle function” and a primitive for the Liouville form of the network  $\mathbb{S}_0^\xi$ , as defined in Section 11.1.

$\mathcal{P}3$ . The curvature of  $\mathbb{S}_0^\xi$  is bounded by  $C/\sqrt{\xi}$  and  $\mathbb{S}_0^\xi \rightarrow \mathbb{S}_0$  in  $C_{\text{loc}}^1(\mathbb{R}^2 \setminus \{0\})$ , as  $\xi \rightarrow 0$ .

$\mathcal{P}4$ . The connected components of  $P \cap A(r_0\sqrt{2\xi}, 4)$  are in one-to-one correspondence with the connected components of  $\mathbb{S}_0^\xi \cap A(r_0\sqrt{2\xi}, 4)$  and there is a constant  $D_3$ , independent of  $\xi$ , such that the functions  $u_{j,\xi}$  satisfy

$$|u_{j,\xi}(\ell)| + \ell|u'_{j,\xi}(\ell)| + \ell^2|u''_{j,\xi}(\ell)| \leq D_3\left(\ell^2 + \sqrt{2\xi} e^{-\ell^2/4\xi}\right),$$

for every  $\ell \in [r_0\sqrt{2\xi}, 4]$ .

$\mathcal{P}5$ . The sequence of rescaled networks  $\tilde{\mathbb{S}}_0^\xi = \mathbb{S}_0^\xi/\sqrt{2\xi}$  converges in  $C_{\text{loc}}^{1,\alpha}(B_{r_0}(0))$  to  $\mathbb{E}$ , for  $\alpha \in (0, 1)$ , as  $\xi \rightarrow 0$ .

Without loss of generality we can also assume that locally

$$\lim_{\xi \rightarrow 0}(\tilde{\theta}_0^\xi + \tilde{\beta}_0^\xi) = 0,$$

where  $\tilde{\theta}_0^\xi$  and  $\tilde{\beta}_0^\xi$  are relative to  $\tilde{\mathbb{S}}_0^\xi$ .

Let  $\mathbb{S}_t^\xi$ , for  $t \in [0, T_\xi)$ , be the maximal smooth curvature flow starting at the initial network  $\mathbb{S}_0^\xi$ , obtained by Theorem 6.8 (or Theorem 4.18 if  $\mathbb{S}_0$  is smooth) and let

$$\Theta_{x_0, t_0}^\xi(t) = \int_{\mathbb{S}_t^\xi} \rho_{x_0, t_0}(\cdot, t) ds$$

be the Gaussian density function with respect to the flow  $\mathbb{S}_t^\xi$ .

We fix  $\varepsilon_0 > 0$  such that  $3/2 + \varepsilon_0 < \Theta_{S^1}$ . The main estimate, which will imply short time existence, is given by the following proposition.

**Proposition 11.19.** *There are constants  $\xi_1$ ,  $\delta_1$  and  $\eta_1$  depending on  $D_1$ ,  $D_2$ ,  $D_3$ ,  $\mathbb{E}$ ,  $r_0$  and  $\varepsilon_0$ , such that if*

$$t \leq \delta_1, \quad r^2 \leq \eta_1^2 t, \quad \text{and} \quad \xi \leq \xi_1,$$

then,

$$\Theta_{x, t+r^2}^\xi(t) \leq 3/2 + \varepsilon_0,$$

for every  $x \in B_1(0)$ .

We will sketch the proof after showing how this implies Theorem 11.1.

*Proof of Theorem 11.1.* Considering the smooth curvature flows  $\mathbb{S}_t^\xi$  in the time interval  $[0, T_\xi]$ , for some  $T_\xi > 0$ , discussed above, we now aim to show that there exists  $T > 0$  such that  $T_\xi \geq T$ , for all  $\xi \in (0, \xi_1)$  and that there are interior estimates on the curvature and all its higher derivatives for all positive times, independent of  $\xi \in (0, \xi_1)$ .

By [47, Theorem 1.5], there exists  $\varepsilon > 0$  such that if  $\mathbb{S}_0^\xi$  can be written with respect to suitably chosen coordinate system as a graph with a small gradient in a ball  $B_R(x)$ , then  $\mathbb{S}_t^\xi$  remains a graph in this coordinate system in  $B_{\varepsilon R}(x)$  with small gradient, for  $t \in [0, \varepsilon R^2]$ . Combining this fact with the interior estimates of Ecker–Huisken in [26] for the curvature and its higher derivatives, we can choose a parametrization of the evolving network and a smooth family of points  $\bar{P}_j^\xi \in \mathbb{S}_t^\xi$  in the annulus  $B_{1/2}(0) \setminus B_{1/3}(0)$  along each curve corresponding to  $P_j$ , for  $j = 1, \dots, n$ , such that

$$\partial_s^l \lambda(\bar{P}_j^\xi, t) = 0 \quad \text{and} \quad |\partial_s^l k(\bar{P}_j^\xi, t)| \leq C_l,$$

for all  $l \geq 0$  with constants  $C_l$  independent of  $\xi$  for  $0 \leq t < \min\{T_\xi, \delta\}$ , where  $\delta > 0$  does not depend on  $\xi$ . Then, Corollary 5.11 gives estimates on the curvature and its derivatives, independent of  $\xi$  and  $t$ , on  $\mathbb{S}_t^\xi \setminus B_{1/2}(0)$ , for  $t \in (0, \min\{T_\xi, \delta\})$  (possibly taking a smaller  $\delta > 0$ ).

To get the desired estimates on  $\mathbb{S}_t^\xi \cap B_{1/2}(0)$  we now apply Proposition 11.19 and Theorem 9.3. Let  $\xi_1, \delta_1, \eta_1$  be given by Proposition 11.19. If we choose  $0 < t_0 < \min\{T_\xi, \delta_1, \delta, 1/2\}$  and  $x_0 \in B_{1/2}(0)$ , Proposition 11.19 implies that if  $\xi < \xi_1$ , we have

$$\Theta_{x, t+r^2}^\xi(t) \leq 3/2 + \varepsilon_0,$$

for all  $x \in B_1(0)$ ,  $t \in (0, t_0)$  and  $r^2 \leq \eta_1^2 t$ . In particular, we see that if  $\bar{t} \in (t_0/2, t_0)$ , choosing  $r^2 \leq \frac{\eta_1^2 t_0}{2(1+\eta_1^2)}$  and setting  $t = \bar{t} - r^2$ , we have  $t < t_0 \leq \delta_1$  and  $r^2 \leq \eta_1^2 t$ . Hence, the above estimate holds and it can be equivalently written as

$$\Theta_{x, \bar{t}}^\xi(\bar{t} - r^2) \leq 3/2 + \varepsilon_0,$$

for such pairs  $(\bar{t}, r)$ . Letting  $\rho = \sqrt{t_0/2}$  (notice that  $B_\rho(x_0) \subset B_1(0)$ ), such estimate holds for all  $(x, \bar{t}) \in B_\rho(x_0) \times (t_0 - \rho^2, t_0)$  and  $r \leq \frac{\eta_1^2}{\sqrt{1+\eta_1^2}} \rho$ . Hence, by Theorem 9.3 with  $\sigma = 1/2$ , there exists a constant  $C$ , depending only on  $\varepsilon_0$  and  $\eta_1$  (by property  $\mathcal{P}1$  above, the length ratios are uniformly bounded) such that

$$|k^\xi(x, \bar{t})| \leq C/\sqrt{t_0},$$

for every  $\bar{t} \in (t_0/8, t_0)$  and  $x \in \mathbb{S}_{\bar{t}}^\xi \cap B_{\sqrt{t_0/8}}(0)$ . Sending  $\bar{t} \rightarrow t_0$ , we get

$$|k^\xi(x_0, t_0)| \leq C/\sqrt{t_0}.$$

Hence, by the arbitrariness of  $x_0$ , this estimates holds for all  $x_0 \in \mathbb{S}_{t_0}^\xi \cap B_{1/2}(0)$  and  $t_0$  small enough, together with the corresponding estimates on all higher derivatives. Moreover, by the second point of Remark 9.4, there is a constant  $C_1 > 0$ , depending only on  $\varepsilon_0$  and  $\eta_1$ , such that the length of the shortest curve of  $\mathbb{S}_{t_0}^\xi$  is bounded from below by  $C_1 \sqrt{t_0}$ . By the arbitrariness of the choice, these estimates hold for every  $t_0 > 0$  small enough.

Together with the estimates on  $\mathbb{S}_t^\xi \setminus B_{1/2}(0)$  for every  $t \in (0, \min\{T_\xi, \delta\})$ , this implies that  $T_\xi \geq T$ , for some  $T > 0$ , for every  $\xi \leq \xi_1$ . By the estimates on the curvature, which are independent of  $\xi$ , we can then take a subsequential limit of the flows  $\mathbb{S}_t^\xi$  on  $[0, T)$ , as  $\xi \rightarrow 0$ , to obtain a smooth limit curvature flow  $\mathbb{S}_t$  in a positive time interval, starting from the non-regular network  $\mathbb{S}_0$ .

Notice that, by [47, Theorem 1.5] and the interior estimates of Ecker–Huisken, away from any multi-point, the flow  $\mathbb{S}_t$  attains the initial network  $\mathbb{S}_0$  in  $C^2$  (or in the class of regularity of  $\mathbb{S}_0$ , if it is better than  $C^2$  away from the multi-point).

Furthermore, by the above estimate on the curvature and Theorem 9.3, we have

$$|k(x, t)| \leq C/\sqrt{t},$$

for every  $x \in \mathbb{S}_t$ . The estimate on the length of the shortest curve passes to the limit as well.  $\square$



*Remark 11.20.* The conclusions of Theorem 11.1 also hold if the initial network  $\mathbb{S}_0$  is a  $C^1$  non-regular network, smooth away from the multi-points where the exterior unit tangent vectors of the concurring curves are mutually distinct and the curvature is of order  $o(1/r)$ , where  $r$  is the distance from the set of the multi-points of  $\mathbb{S}_0$ .

The modifications in the proof are not difficult, but not completely trivial. The details of such result will appear elsewhere.

We will now give a sketch of the proof of Proposition 11.19. Since the estimates are rather technical we only outline it and refer the interested reader to [47], however, we want to underline the main three steps of the proof.

**Step 1.** *Estimates far from the origin and for short time.*

The following estimates are a direct consequence of Huisken's monotonicity formula (7.1): the first one says that the flow is well controlled at a point  $x$  away from the origin up to a time proportional to  $|x|^2$ . This follows by observing that in the annulus  $A(K_0\sqrt{2\xi}, 1)$ , where  $K_0$  is sufficiently large, the initial network  $\mathbb{S}_0^\xi$  is close to the collection of halflines  $P$  for all  $0 < \xi \leq \xi_1$ . Even more, for  $1 \geq |x| \geq K_0\sqrt{2(\xi+t)}$  we see that in  $B_{(K_0/2)\sqrt{2(\xi+t)}}(x)$  the initial network is  $C^1$ -close to a unit density line. By the monotonicity formula this gives a control up to time  $t$ .

The second one shows that if we glue in the regular expander at scale  $\xi$ , then we get control in  $t$  up to a time proportional to  $\xi$ . This estimate follows from observing that scaling the initial network  $\mathbb{S}_0^\xi$  by  $1/\sqrt{2\xi}$ , each point on the network is uniformly  $C^1$ -close, in a ball of fixed size, either to a unit density line, or to a standard triod. The estimate then follows from the monotonicity formula.

For details of the proof see [47, Lemma 5.2].

**Lemma 11.21.**

- (Far from origin estimate) *There are  $\delta_1, K_0 > 0$  such that if  $r^2 \leq t \leq \delta_1$ , then*

$$\Theta_{x,t+r^2}^\xi(t) \leq 3/2 + \varepsilon_0,$$

*for every  $x$  with  $1 \geq |x| \geq K_0\sqrt{2(\xi+t)}$ .*

- (Short time estimate) *There are  $\xi_1, q_1 > 0$  such that if  $\xi \leq \xi_1$ ,  $r^2, t \leq q_1\xi$ , then*

$$\Theta_{x,t+r^2}^\xi(t) \leq 3/2 + \varepsilon_0,$$

*for every  $x \in B_1(0)$ .*

It is convenient to introduce a rescaling of the flow which makes the expander “stationary”. We set (see property P5 above)

$$\tilde{\mathbb{S}}_t^\xi = \frac{\mathbb{S}_t^\xi}{\sqrt{2(\xi+t)}},$$

and let

$$\tilde{\Theta}_{x_0,t_0}^\xi(t) = \int_{\tilde{\mathbb{S}}_t^\xi} \rho_{x_0,t_0}(\cdot, t) ds.$$

Notice that

$$\Theta_{x_0,t+r^2}^\xi(t) = \tilde{\Theta}_{\frac{x_0}{\sqrt{2(\xi+t)}}, t + \frac{r^2}{2(\xi+t)}}^\xi(t). \quad (11.3)$$

*Remark 11.22.*

1. It follows from the second estimate in Lemma 11.21 that we need only to prove Proposition 11.19 when  $t \geq q_1\xi$ .
2. By formula (11.3) and the previous point, it suffices to find  $\xi_1, \delta_1$  and  $\eta_1$  such that for every  $\xi \leq \xi_1$ ,  $q_1\xi \leq t \leq \delta_1$ ,  $r^2 \leq \eta_1^2$  and  $y$  with  $|y| \leq 1/\sqrt{2(\xi+t)}$ , we have

$$\tilde{\Theta}_{y,t+r^2}^\xi(t) \leq 3/2 + \varepsilon_0.$$

3. We set  $\eta_1^2 = q_1/(2(q_1 + 1))$ . The second estimate in Lemma 11.21 implies that for  $\xi \leq \xi_1$ ,  $t \leq q_1\xi$  and  $r^2 \leq \eta_1^2$  we have

$$\tilde{\Theta}_{y,t+r^2}^\xi(t) \leq 3/2 + \varepsilon_0,$$

for every  $|y| \leq 1/\sqrt{2(\xi+t)}$ .

The first estimate in Lemma 11.21 implies that for  $r^2 \leq \eta_1^2$ ,  $\xi \leq \xi_1$  and  $q_1\xi \leq t \leq \delta_1$ ,

$$\tilde{\Theta}_{y,t+r^2}^\xi(t) \leq 3/2 + \varepsilon_0,$$

for every  $y$  with  $K_0 \leq |y| \leq 1/\sqrt{2(\xi+t)}$ .

**Step 2.** *Controlling the asymptotic behavior of  $\tilde{\mathbb{S}}_t^\xi$ .*

By some rather delicate estimates, but which only use the asymptotics  $\mathcal{P}4$  and again the monotonicity formula, one can show that the following holds (see Lemma [47, Lemma 5.4]). It is important here that  $r_1$  does not depend on  $\nu$ .

**Lemma 11.23** (Proximity to  $P$ ). *There are constants  $C_1$  and  $r_1$  such that, for every  $\nu > 0$ , we can find  $\xi_2, \delta_2 > 0$  such that the following holds. If  $\xi \leq \xi_2, t \leq \delta_2$  and  $r \leq 2$ , then*

$$\text{dist}(y, P) \leq \nu + C_1 e^{-|y|^2/C_1} \text{ if } y \in \tilde{\mathbb{S}}_t^\xi \cap A(r_1, (\xi+t)^{-1/8}),$$

and

$$\tilde{\Theta}_{y,t+r^2}^\xi(t) \leq 1 + \varepsilon_0/2 + \nu \text{ if } y \in A(r_1, (\xi+t)^{-1/8}),$$

where  $A(r_1, (\xi+t)^{-1/8})$  is the annulus  $B_{(\xi+t)^{-1/8}}(0) \setminus B_{r_1}(0)$ .

The next step is to combine these estimates with the uniqueness of the regular expander in its topological class, given by Theorem 11.16, and a compactness argument (see [47, Corollary 4.6]) to show the following:

**Lemma 11.24.** *Let  $C_1$  and  $r_1$  be the constants given by Lemma 11.23 and let  $\mathbb{E}$  be a regular expander. Set  $r_2 = \max\{r_0, r_1, 1\}$ ,  $R = \sqrt{1 + 2q_1}K_0 + r_2$ . Then there exist  $R_1 \geq R$ ,  $\varrho, \nu > 0$  such that if  $\mathbb{S}$  is a regular network with controlled length ratios such that:*

1.  $\int_{\mathbb{S} \cap B_{R_1}(0)} |\underline{k} - x^\perp|^2 ds \leq \varrho$ ,
2.  $\mathbb{S}$  and  $\mathbb{E}$  are in the same topological class (see Definition 11.15),

then  $\mathbb{S}$  must be  $\varepsilon$ -close in  $C^{1,\alpha}(B_{R_1}(0))$  to  $\mathbb{E}$ , for a fixed  $\alpha \in (0, 1/2)$  and a suitably small  $\varepsilon > 0$ , depending on  $\mathbb{E}$ .

Notice that  $\varepsilon$  has to be chosen sufficiently small, so that the monotonicity formula guarantees a control of the Gaussian densities for a network  $C^{1,\alpha}$ -close to  $\mathbb{E}$ .

**Step 3.** *Application of the expander monotonicity formula.*

The next lemma is essential to prove Proposition 11.19. Its content is that the proximity of  $\tilde{\mathbb{S}}_t^\xi$  to the self-similarly expanding curvature flow generated by  $\mathbb{E}$  can be controlled in an integral sense. This is the only point where the expander monotonicity formula is used.

We notice that by property  $\mathcal{P}5$  above, we have that  $\tilde{\mathbb{S}}_0^\xi = \sqrt{2\xi}\mathbb{S}_0^\xi \rightarrow \mathbb{E}$  in  $C_{\text{loc}}^{1,\alpha}(B_{r_0}(0))$ , as  $\xi \rightarrow 0$ , and recall that the rescaled quantity

$$\tilde{\alpha}_t^\xi = \tilde{\beta}_t^\xi + \tilde{\theta}_t^\xi,$$

of the expander monotonicity formula, converges locally to zero along this limit. Localizing the expander monotonicity formula (Lemma 11.9), choosing  $(x_0, t_0)$  appropriately and estimating carefully, one arrives at the following (see [47, Lemma 5.6]). Choose  $a > 1$  such  $(1 + 2q_1)/a > 1$  and set  $q = q_1/a$ .

**Lemma 11.25.** *There are constants  $\delta_0$  and  $\xi_0$  such that for every  $\xi \leq \xi_0$  and  $T_0 \in [q\xi, \delta_0]$ , we have*

$$\frac{1}{(a-1)T_0} \int_{T_0}^{aT_0} \int_{\tilde{\mathbb{S}}_t^\xi \cap B_{R_1}(0)} |\underline{k} - x^\perp|^2 ds dt \leq \varrho.$$

Take  $\delta_0, \xi_0$  for which this lemma holds, consider also  $\delta_1, \xi_1$  for which Lemma 11.21 holds and  $\xi_2 = \xi_2(\nu)$ ,  $\delta_2 = \delta_2(\nu)$  given by Lemma 11.23. Set  $\xi_3 = \min\{\xi_0, \xi_1, \xi_2\}$ ,  $\delta_3 = \min\{\delta_0, \delta_1, \delta_2\}$  and then, decrease  $\xi_3$  and  $\delta_3$ , if necessary, so that  $(\xi_3 + \delta_3)^{-1/8} \geq 2R_1$ ,  $q_1\xi_3 \leq \delta_3$ . Having all the constants properly defined, we can now finish the proof. Set

$$T_1 = \sup\{ \tilde{T} \mid \tilde{\Theta}_{x, t+r^2}^\xi(t) \leq 3/2 + \varepsilon_0 \quad \text{for all } x \in B_{K_0}(0), r^2 \leq \eta_1^2, t \leq \tilde{T} \}.$$

It suffices to show that  $T_1 \geq \delta_3$ , for every  $\xi \leq \xi_3$ . The first point of Remark 11.22 implies that  $T_1 \geq q_1\xi$ . Suppose that  $T_1 < \delta_3$  and set  $T_2 = T_1/a$ . Lemma 11.25 implies the existence of  $t_1 \in [T_2, T_1]$  such that

$$\int_{\tilde{\mathbb{S}}_{t_1}^\xi \cap B_{R_1}(0)} |\underline{k} - x^\perp|^2 ds \leq \varrho.$$

One can now check that all the conditions for the previous step are met with  $\mathbb{S}$  being  $\tilde{\mathbb{S}}_{t_1}^\xi$ . Therefore, we obtain that  $\tilde{\mathbb{S}}_{t_1}^\xi$  is  $\varepsilon$ -close in  $C^{1,\alpha}(B_{R_1}(0))$  to  $\mathbb{E}$ . Denote by  $\hat{\mathbb{S}}_l^\xi$ , for  $l \geq 0$ , the curvature flow with initial condition  $\tilde{\mathbb{S}}_{t_1}^\xi$ . A simple computation shows that

$$\hat{\mathbb{S}}_l^\xi = \sqrt{1+2l} \tilde{\mathbb{S}}_{t_1+l\mu^2}^\xi,$$

where  $\mu^2 = 2(\xi + t_1)$ . Since  $\tilde{\mathbb{S}}_{t_1}^\xi$  is  $\varepsilon$ -close in  $C^{1,\alpha}(B_{R_1}(0))$  to  $\mathbb{E}$ , we again use the monotonicity formula to conclude that for every  $l \leq q_1$ , we have

$$\tilde{\Theta}_{x, t_1+l\mu^2+r^2}^\xi(t_1+l\mu^2) = \hat{\Theta}_{x\sqrt{1+2l}, l+r^2(1+2l)}^\xi(l) \leq 3/2 + \varepsilon_0,$$

provided

$$\sqrt{1+2l}|x| \leq R_1 - 1 \quad \text{and} \quad (1+2l)r^2 \leq q_1.$$

Hence, for all  $t_1 \leq t \leq t_1(1+2q_1)$ , there holds

$$\tilde{\Theta}_{x, t+r^2}^\xi(t) \leq 3/2 + \varepsilon_0,$$

for every  $x$  in  $B_{K_0}(0)$  and  $r^2 \leq \eta_1^2$ , which implies that  $T_1 \geq t_1(1+2q_1)$ . This is a contradiction because

$$t_1(1+2q_1) \geq T_2(1+2q_1) = T_1(1+2q_1)/a > T_0.$$

This concludes the proof of Proposition 11.19. □

*Remark 11.26.*

- Putting together Theorem 11.1 and Theorem 6.8 (or Theorem 4.18, if  $\mathbb{S}_0$  is smooth) we have a curvature flow (in the sense of Brakke), which is smooth for every positive time, for every initial  $C^2$  network  $\mathbb{S}_0$  (satisfying the hypothesis that at every multi-point the exterior unit tangent vectors of the concurring curves are mutually distinct – see anyway Remark 11.4).
- Notice that in the above proof, we do not perform the “gluing in” construction at the regular 3-points of the initial network. Hence, since the approximating flows are based on Theorem 6.8 (or Theorem 4.18, if  $\mathbb{S}_0$  is smooth), the convergence of  $\mathbb{S}_t$  to  $\mathbb{S}_0$ , as  $t \rightarrow 0$ , locally around a regular 3-point of  $\mathbb{S}_0$  is the one given by such theorems. Clearly, one could apply the “gluing in” procedure also at the regular 3-points (in such case the regular expander  $\mathbb{E}$  to be “glued in” is simply a standard triod). Then, a natural question is if the convergence of  $\mathbb{S}_t \rightarrow \mathbb{S}_0$  locally around such regular 3-point is at least  $C^1$  or better (depending on the regularity of  $\mathbb{S}_0$  and the level of compatibility conditions it satisfies) and what is the relation between this curvature flow and the one instead obtained by Theorem 6.8.
- In the special situation when we want to use Theorem 11.1 to “restart” a limit non-regular network  $\mathbb{S}_T$ , after a singularity at time  $T$  (if possible), far from its multi-points  $O^1, O^2, \dots, O^m$  such network is smooth, hence,  $\mathbb{S}_t \rightarrow \mathbb{S}_T$  in  $C_{\text{loc}}^\infty(\mathbb{R}^2 \setminus \{O^1, O^2, \dots, O^m\})$ , as  $t \rightarrow T$ .

*Remark 11.27.*

- As we said, given the set  $P$ , finite union of  $n$  halflines for the origin, with  $n > 3$ , there could be more than one regular expander asymptotic to  $P$ , even restricting ourselves to the class of the tree-like ones (see Figure 28, for instance). In some special situations, there is a unique tree-like, connected, regular expander, notably when  $P$  is a standard cross, see Corollary 11.17 (composed by four halflines from the origin with opposite directions pairs and forming angles of 120/60 degrees between them), generated by the exterior unit tangents of the four concurring curves at the 4-point which arises as the collapse with bounded curvature of a curve in the “interior” of  $\mathbb{S}_t$ , as  $t \rightarrow T$ , described in Proposition 10.14.

One would like to have, at least for the “generic” family of halflines  $P$ , a sort of “selection principle” to choose the “best” regular expander  $\mathbb{E}$  to “glue it in” at a multi-point with more than 3 concurring curves, in the above procedure.

- A simple uniqueness statement (which can hold, by what we said, only for a “generic” initial network) for the curvature flow obtained by Theorem 11.1 is missing at the moment, in general.

**Open Problem 11.28.** There exists, at least for a “generic” family of four or more halflines for the origin  $P$ , a “selection principle” to choose the “best” regular expander  $\mathbb{E}$  asymptotic to  $P$ , in order to “glue it in” at a multi-point  $O^i$  of  $\mathbb{S}_0$ , with  $P$  generated by the exterior unit tangent vectors of the curves concurring at  $O^i$ , to perform the procedure of Theorem 11.1?

**Open Problem 11.29.** In what class of curvature flows, for a “generic” initial non-regular network  $\mathbb{S}_0$ , the flow given by Theorem 11.1 is unique?

## 12 Restarting the flow after a singular time

By means of the analysis of Section 10.4 and the description of the limit network  $\mathbb{S}_T$  at a singular time in Theorems 10.37 and 10.38, we can continue the flow, applying the “restarting” Theorem 11.1 (and its extension, see Remark 11.20). We then have an “extended” curvature flow for some positive time  $T' > T$  (if we are not in some of the situations, discussed in Section 10.4, when the flow “naturally ends” – for instance, if the whole network collapses and vanishes, as  $t \rightarrow T$ ), which is a Brakke flow (possibly without equality, see Remark 11.3) in the time interval  $(0, T')$  and a smooth curvature flow in  $(0, T) \cup (T, T')$ .

The passage through a singularity when (locally) a single curve vanishes, and two triple junctions collapse forming a 4-point in  $\Omega$  is particularly interesting, as this type of singularity is the only possible one for the motion of a tree-like network. We call this change of the structure of the network a “standard transition” (see Figures 29, 30).

We recall that the curvature stays uniformly bounded for  $t \leq T$ , while it is of order  $C/\sqrt{T-t}$  as  $t \rightarrow T$  (and the “new” segment has length of order  $\sqrt{T-t}$ ).

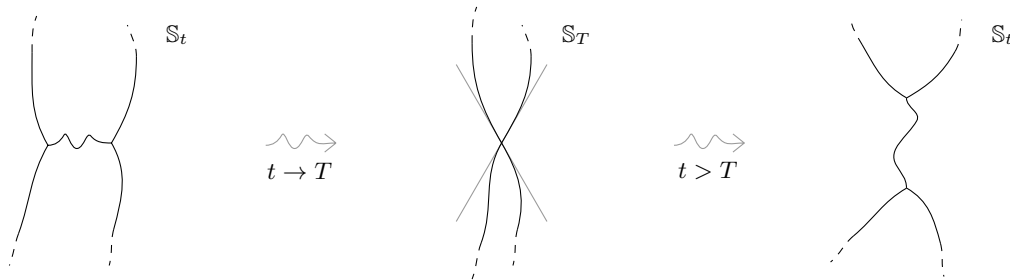


Figure 29: The local description of a “standard” transition.

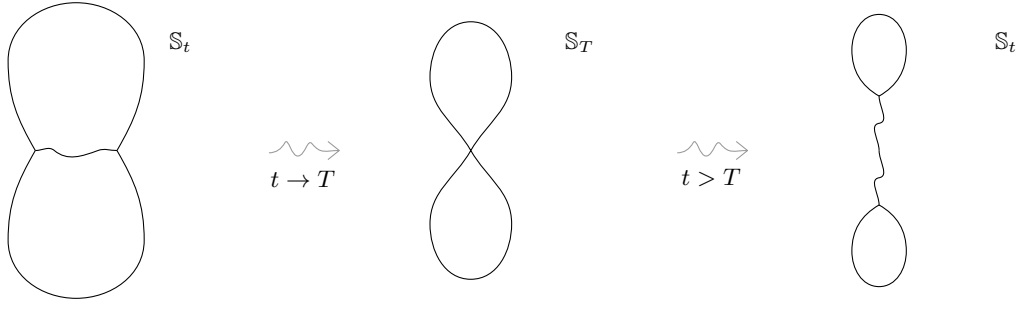


Figure 30: A “standard” transition for a  $\Theta$ -shaped network (double cell).

We remark that such transition, passing by  $S_T$ , is not symmetric: when  $S_t \rightarrow S_T$ , as  $t \rightarrow T^-$ , the exterior unit tangent vectors, hence the four angles between the curves, are continuous, while when  $S_t \rightarrow S_T$ , as  $t \rightarrow T^+$ , there is a “jump” in such angles, precisely, there is an instantaneous “switch” between the angles of 60 degrees and the angles of 120 degrees at time  $T$ .

*Remark 12.1.* Since there is a single expander “coming out” from the cone of the inner unit tangent vectors generated by the four concurring curves, we expect that the restarting by means of Theorem 11.1 produces a unique evolution (see Problem 11.18).

Coming back to the general situation, we list a series of facts when passing through a singularity.

- The total length of the evolving network  $S_t$  is non increasing and continuous for every  $t \in (0, T')$ . Hence, as a Brakke flow in the time interval  $[0, T')$  it does not suffer of the phenomenon of “sudden mass loss” (see [13] and the recent work [49]).
- For every  $x_0 \in \mathbb{R}^2$  and  $t_0 \in (0, +\infty)$ , the Gaussian density function  $\Theta_{x_0, t_0}(t) : [0, \min\{t_0, T'\}) \rightarrow \mathbb{R}$  is still non increasing. The same for the entropy of  $S_t$ , see formula (8.5).
- The uniform bound on length ratios survives the “restarting” procedure with the same constant.

These points follow easily by the (weak) continuity of the Hausdorff measures  $\mathcal{H}^1 \llcorner S_t$ , see Remarks 10.36 and 11.2 (it is clear in the case of a standard transition).

- By the construction in the “restarting” Theorem 11.1, no new regions are created passing a singularity, their total number is non increasing. In particular, a tree remains a tree after restarting (even if its “structure” changes).
- The number of curves of the network is not increasing. To be more precise, if at least a region vanishes the total number of curves decreases at least of three. In a standard transition it remains the same.
- The number of triple junctions of the network is non increasing. To be more precise, if at least a region vanishes the total number of triple junctions decreases at least of two. In a standard transition it remains the same.

The fact that no new regions arise follows by the fact that we “desingularise” a multi-point, in Theorem 11.1, by gluing in a tree-like, connected, regular expander (which is an a priori choice, see Remark 11.5). In doing that, by means of Euler’s formula for trees, we can see that if the multi-point has order  $n$ , being the number of the regions equal to  $n$ , the number of triple junctions we will have in the restarted network in place of the single multiple junction is equal to  $n - 2$  and the number of curves is  $2n - 3$ .

It is then easy to check the above statements, if only one bounded region is collapsing, since it must be bounded by  $n$  curves. If instead a group of regions is collapsing, we can get the conclusion by applying the same argument to the bounded “macro-region” that we obtain considering their union, which will be bounded by a piecewise smooth loop (in a way, we are “forgetting” the interior curves to such “macro-region” which will anyway be “lost” in the collapse).

Clearly, all these facts say that, in a sense, the “topological complexity” of the network is “non increasing” passing through a singular time.

We finally mention here that also the bound on the “embeddedness measure”  $E(t)$ , which we will introduce in Section 14, survives the “restarting” procedure.

### 13 Long time behavior

Since we can repeat the restarting procedure at every singular time, either the flow naturally ends at some time  $\hat{T}$  (for instance, if the whole network collapses and vanishes, as  $t \rightarrow \hat{T}$ ) or we found ourselves in some of the situations described in Section 10.4 where we have to decide how to continue the flow (related to the behavior at the boundary of  $\Omega$ ), or we have an increasing sequence of singular-restarting times  $T_i$  for the evolution of the network  $\mathbb{S}_t$ . In this latter case it follows by the “topological” conclusions in the previous section that among these times  $T_i$ , the number of the ones such that we have a non-standard transition is actually finite and depending only on  $\mathbb{S}_0$  (indeed if a transition is non-standard, then at least one region vanishes during the transition and  $\mathbb{S}_0$  can have only a finite number of regions). Instead, we cannot conclude the same for the number of standard transitions that a priori could be infinite. Even worse, notice that Theorem 11.1 does not give any estimate on the (short) time of existence of the restarted flow, which means that we are not able to say in general if and when another singularity could appear after the restarting time. Hence, in particular, we are also not able to exclude that the singular times (associated to standard transitions) actually may “accumulate”, not even for a tree-like network when all the possible singularities are exactly standard transitions.

The following figures shows some examples of these (maybe) possible situations.

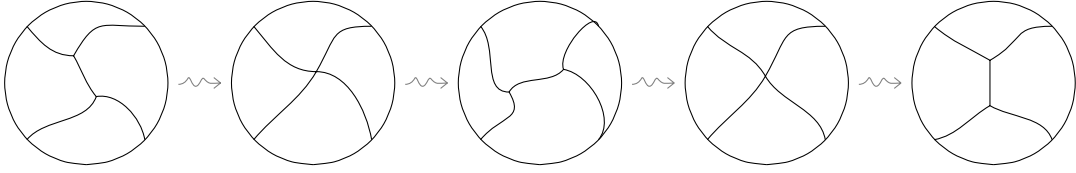


Figure 31: A tree-like network with four fixed end-points switching between its two possible topological classes.

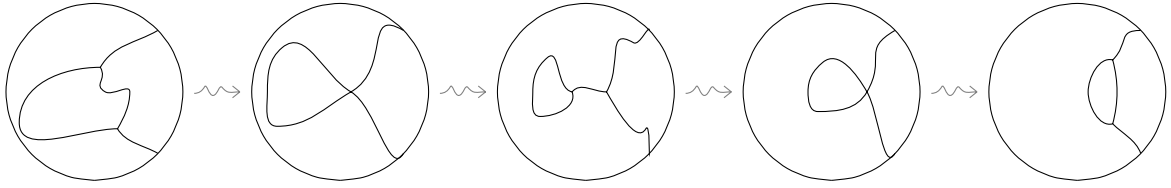


Figure 32: Standard transitions switching a lens-shaped network to an “island-shaped” (with a bridge) one and viceversa.

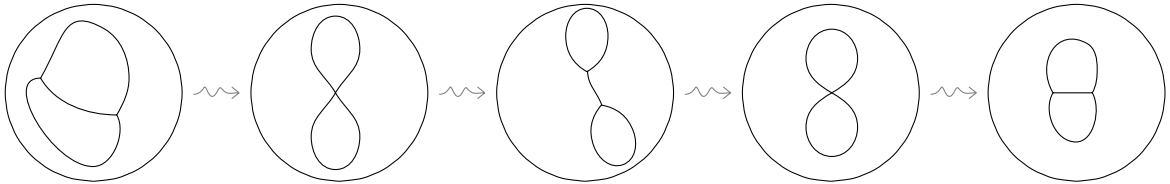


Figure 33: Switching by standard transitions of a  $\Theta$ -shaped network to an “eyeglasses-shaped” one and viceversa.

In all these examples (where there is a sort of “duality” between the two involved networks) we do not know if this kind of “oscillatory phenomenon” can happen infinite times (possibly even discrete in the case of the tree).

**Open Problem 13.1.** Let us assume that the “boundary” curves do not collapse during the flow.

- The number of singular times is finite?
- The singular times can accumulate?
- If the flow does not fully vanish at some time (for instance, if some boundary curves are present), it can be defined for every positive time?

*Remark 13.2.* The last question asks whether we can always find a “well-behaved” limit network  $\mathbb{S}_T$ , even when the singular times  $T_i$  accumulate at some  $\hat{T} > 0$ , in order to possibly restart again the flow with Theorem 11.1 or some extension of it. Indeed, by restarting the flow at every singularity we can define an *extended* curvature flow of networks on a time interval  $[0, \hat{T})$  where at the maximal time  $\hat{T}$ , if it is finite, either the whole network vanishes, or there is an accumulation of singular times. This extended curvature flow is a Brakke flow, by Theorem 11.1, and actually only times when a standard transition happens can accumulate at  $\hat{T}$ .

We also mention that it would be quite interesting to compare this extended curvature flow of networks with the globally defined one in the recent paper [49] by L. Kim and Y. Tonegawa.

We now discuss the long time behavior of the curvature flow of a regular network, assuming that there is no accumulation of the singular times or that the flow does not have anymore singularities after some time  $T_\infty$ . We see immediately that this latter case can only happen for tree-like networks.

**Proposition 13.3.** Let  $\mathbb{S}_t$  be a smooth curvature flow in  $[0, T)$  of a network that has at least one loop  $\ell$  of length  $L(t)$ , enclosing a region of area  $A(t)$ , composed by  $m$  curves with  $m < 6$ , where  $T$  is the maximal time of smooth evolution. Then,  $T \leq \frac{3A(0)}{(6-m)\pi}$  and the equality holds if and only if  $\lim_{t \rightarrow T} A(t) = 0$ . Moreover, if  $\lim_{t \rightarrow T} L(t) = 0$ , then  $\lim_{t \rightarrow T} \int_{\mathbb{S}_t} k^2 ds = +\infty$ .

*Proof.* Integrating in time the equation (8.2), we have

$$A(t) - A(0) = \left(-2\pi + m \left(\frac{\pi}{3}\right)\right) t.$$

Therefore,  $T \leq \frac{3A(0)}{(6-m)\pi}$ , with equality if and only if  $\lim_{t \rightarrow T} A(t) = 0$ .

Suppose now that  $\lim_{t \rightarrow T} L(t) = 0$ . Then, we necessarily have  $T = \frac{3A(0)}{(6-m)\pi}$  and  $\lim_{t \rightarrow T} A(t) = 0$ . Combining equation (8.2) and Hölder inequality, we get

$$\left| -2\pi + m \left(\frac{\pi}{3}\right) \right| = \left| \frac{dA(t)}{dt} \right| = \left| \int_{\ell_t} k ds \right| \leq (L(t))^{\frac{1}{2}} \left( \int_{\ell_t} k^2 ds \right)^{\frac{1}{2}},$$

which gives

$$\int_{\mathbb{S}_t} k^2 ds \geq \int_{\ell_t} k^2 ds \geq \frac{(6-m)^2 \pi^2}{9L(t)}.$$

Since  $\lim_{t \rightarrow T} L(t) = 0$ , it follows that  $\lim_{t \rightarrow T} \int_{\mathbb{S}_t} k^2 ds = +\infty$ . □

*Remark 13.4.*

1. If a loop is composed by six or more curves, then, by equation (8.2), either the enclosed area remains constant or increases during the evolution.
2. The previous proposition clearly does not exclude the possibility that a singularity appears at a time  $T < \frac{3A(0)}{(6-m)\pi}$ .
3. We expect that, if  $T = \frac{3A(0)}{(6-m)\pi}$ , then  $\lim_{t \rightarrow T} L(t) = 0$  and  $\lim_{t \rightarrow T} \int_{\mathbb{S}_t} k^2 ds = +\infty$ .



For a general network, since we assumed that there is no accumulation of the singular times, if the boundary curves do not collapse, we cannot exclude that there could be an infinite sequence of standard transitions with some loops present and never collapsing. We anyway conjecture that at some time any evolving network either vanishes or it becomes a tree. This clearly follows if the total number of singular times is finite and it is actually sustained by computer simulations.

We now deal with tree-like networks that after some time have no more singularities.

**Proposition 13.5.** *Suppose that  $\mathbb{S}_t$  is a smooth curvature flow in  $[0, +\infty)$  of a tree-like network. Then, for every sequence of times  $t_i \rightarrow \infty$ , there exists a (not relabeled) subsequence such that the evolving networks  $\mathbb{S}_{t_i}$  converge in  $C^{1,\alpha} \cap W^{2,2}$ , for every  $\alpha \in (0, 1/2)$ , to a possibly degenerate regular network with zero curvature, that is, “stationary” for the length functional, as  $i \rightarrow \infty$ .*

*Proof.* From equation (5.2), we have the estimate

$$\int_0^{+\infty} \int_{\mathbb{S}_t} k^2 ds dt \leq L(0) < +\infty. \quad (13.1)$$

Then, suppose by contradiction that for a sequence of times  $t_j \nearrow +\infty$  we have  $\int_{\mathbb{S}_{t_j}} k^2 ds \geq \delta$  for some  $\delta > 0$ . By the following estimate, which is inequality (10.4) in Lemma 10.23,

$$\frac{d}{dt} \int_{\mathbb{S}_t} k^2 ds \leq C \left( 1 + \left( \int_{\mathbb{S}_t} k^2 \right) \right)^3,$$

holding (in the case of fixed end-points) with a uniform constant  $C$  independent of time, we would have  $\int_{\mathbb{S}_{\tilde{t}}} k^2 ds \geq \frac{\delta}{2}$ , for every  $\tilde{t}$  in a uniform neighborhood of every  $t_j$ . This is clearly in contradiction with the estimate (13.1). Hence,  $\lim_{t \rightarrow +\infty} \int_{\mathbb{S}_t} k^2 ds = 0$  and, consequently, for every sequence of times  $t_i \rightarrow +\infty$ , there exists a subsequence (not relabeled) such that the evolving networks  $\mathbb{S}_{t_i}$  converge in  $C^{1,\alpha} \cap W^{2,2}$ , for every  $\alpha \in (0, 1/2)$ , to a possibly degenerate regular network with zero curvature, as  $i \rightarrow \infty$ .  $\square$

*Remark 13.6.* The previous proposition shows that, up to subsequences, the sequence of evolving networks  $\mathbb{S}_{t_i}$  converge, as  $t_i \rightarrow +\infty$ , to a “stationary” network for the length functional (which is not necessarily a global minimum). We underline that a priori such a stationary network could be degenerate, that is, in taking the limit of  $\mathbb{S}_{t_i}$  when  $t_i \rightarrow T = +\infty$ , one or more curves could collapse, moreover, it could be non-embedded, with multiplicity greater than one.

*Remark 13.7.* If we do not assume that the number of singularities is finite and/or that the network becomes a tree, but only that the flow exists for every  $t \in [0, +\infty)$ , being globally a Brakke flow (see the previous section), inequality (13.1) still holds (by the defining formula (6.1)) and we can always find a sequence of networks  $\mathbb{S}_{t_i}$  converging in  $C^{1,\alpha} \cap W^{2,2}$ , for every  $\alpha \in (0, 1/2)$ , to a possibly degenerate regular network with zero curvature, as  $i \rightarrow \infty$ . As before, such limit network could be non-embedded.

It is natural to ask ourselves if actually the full sequence  $\mathbb{S}_t$  converges to a limit network as  $t \rightarrow +\infty$ . Moreover, we also expect that such limit network is embedded and that the tree-like hypothesis is actually superfluous.

#### Open Problem 13.8.

- In the hypotheses of Proposition 13.5, the whole sequence of networks  $\mathbb{S}_t$  converge in  $C^{1,\alpha} \cap W^{2,2}$ , for every  $\alpha \in (0, 1/2)$ , to a possibly degenerate regular network with zero curvature, as  $t \rightarrow +\infty$ ?
- The same questions for the general situation described in Remark 13.7.

#### Open Problem 13.9.

- In Proposition 13.5 the tree-like hypothesis can be removed?
- The limit network is embedded?

## 14 An isoperimetric estimate

Given the smooth flow  $\mathbb{S}_t = F(\mathbb{S}, t)$ , we take two points  $p = F(x, t)$  and  $q = F(y, t)$  belonging to  $\mathbb{S}_t$ . A couple  $(p = F(x, t), q = F(y, t))$  is in the class  $\mathfrak{A}$  of the admissible ones if the segment joining  $p$  and  $q$  does not intersect the network  $\mathbb{S}_t$  in other points. Moreover if the network  $\mathbb{S}_t$  has more than one connected component, we take the two points  $p$  and  $q$  in the same connected component.

Given an admissible pair  $(p = F(x, t), q = F(y, t))$  we consider the set of the embedded curves  $\Gamma_{p,q}$  contained in  $\mathbb{S}_t$  connecting  $p$  and  $q$ , forming with the segment  $\overline{pq}$  a Jordan curve. Thus, it is well defined the area of the open region  $\mathcal{A}_{p,q}$  enclosed by any Jordan curve constructed in this way and, for any pair  $(p, q)$ , we call  $A_{p,q}$  the smallest area of all such possible regions  $\mathcal{A}_{p,q}$ . If  $p$  and  $q$  are both points of a set of curves forming a loop, we define  $\psi(A_{p,q})$  as

$$\psi(A_{p,q}) = \frac{A}{\pi} \sin\left(\frac{\pi}{A} A_{p,q}\right),$$

where  $A = A(t)$  is the area of the connected component of  $\Omega \setminus \mathbb{S}_t$  which contains the open segment joining  $p$  and  $q$ .

We consider the function  $\Phi_t : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{R} \cup \{+\infty\}$  as

$$\Phi_t(x, y) = \begin{cases} \frac{|p-q|^2}{\psi(A_{p,q})} & \text{if } x \neq y \text{ and } x, y \text{ are points of a loop;} \\ \frac{|p-q|^2}{A_{p,q}} & \text{if } x \neq y \text{ and } x, y \text{ are not both points of a loop;} \\ 4\sqrt{3} & \text{if } x \text{ and } y \text{ coincide with one of the 3-points } O^i \text{ of } \mathbb{S}; \\ +\infty & \text{if } x = y \neq O^i; \end{cases}$$

where  $p = F(x, t)$  and  $q = F(y, t)$ .

*Remark 14.1.* Following the argument of Huisken in [43], in the definition of the function  $\Phi_t$  we introduce the function  $\psi(A_{p,q})$ , when the two points belong to a loop, because we want to maintain the function smooth also when  $A_{p,q}$  is equal to  $A/2$ .

*Remark 14.2.* As we have already said, if the network  $\mathbb{S}_t$  has more than one connected component, we consider couple of points  $(p, q)$  belonging both to the same connected component. Indeed, if we take the points in two different connected component, the Jordan curve and the area enclosed in it are not defined and the function  $\Phi_t$  has no meaning.

In the following, with a little abuse of notation, we consider the function  $\Phi_t$  defined on  $\mathbb{S}_t \times \mathbb{S}_t$  and we speak of admissible pair for the couples of points  $(p, q) \in \mathbb{S}_t \times \mathbb{S}_t$  instead of  $(x, y) \in \mathbb{S} \times \mathbb{S}$ .

We define  $E(t)$  as the infimum of  $\Phi_t$  between all admissible couple of points  $p = F(x, t)$  and  $q = F(y, t)$ :

$$E(t) = \inf_{(p,q) \in \mathfrak{A}} \Phi_t$$

for every  $t \in [0, T)$ .

We call  $E(t)$  “embeddedness measure”. We underline that similar geometric quantities have already been applied to similar problems in [19, 39, 43].

The following lemma holds, for its proof in the case of a compact network see [19, Theorem 2.1].

**Lemma 14.3.** *The infimum of the function  $\Phi_t$  between all admissible couples  $(p, q)$  is actually a minimum. Moreover, assuming that  $0 < E(t) < 4\sqrt{3}$ , for any minimizing pair  $(p, q)$  we have  $p \neq q$  and neither  $p$  nor  $q$  coincides with one of the 3-points  $O^i(t)$  of  $\mathbb{S}_t$ .*

*Remark 14.4.* In the case of an open network without end-points, since the network is asymptotically  $C^1$ -close to a family of halflines (and during its curvature motion such halflines are fixed), there holds that if the infimum of  $\Phi_t$  is less than a “structural” constant depending only on such halflines, then it is a minimum. By means of such modification to this lemma, all the rest of the analysis of this chapter also holds for the evolution of open networks, we let the details and the easy modifications of the arguments to the reader.

Notice that it follows that the network  $\mathbb{S}_t$  is embedded if and only if  $E(t) > 0$ . Moreover,  $E(t) \leq 4\sqrt{3}$  always holds, thus when  $E(t) > 0$  the two points  $(p, q)$  of a minimizing pair can coincide if and only if

$p = q = O^i(t)$ .

Finally, since the evolution is smooth, it is easy to see that the function  $E : [0, T) \rightarrow \mathbb{R}$  is locally Lipschitz, in particular,  $\frac{dE(t)}{dt} > 0$  exists for almost every time  $t \in [0, t)$ .

If the curvature flow  $\mathbb{S}_t$  has fixed end-points  $\{P^1, P^2, \dots, P^l\}$  on the boundary of a strictly convex set  $\Omega$ , we consider the flows  $\mathbb{H}_t^i$  each obtained as the union of  $\mathbb{S}_t$  with its reflection  $\mathbb{S}_t^{R_i}$  with respect to the end-point  $P^i$ , as we described in the discussion just before Section 10.1.

We underline that this is still a smooth curvature flow (as the compatibility conditions of every order in Definition 4.15 are satisfied by  $\mathbb{S}_t$  at its end-points) without self-intersections, where  $P^i$  is no more an end-point and the number of triple junctions of  $\mathbb{H}_t^i$  is exactly twice the number of the ones of  $\mathbb{S}_t$ .

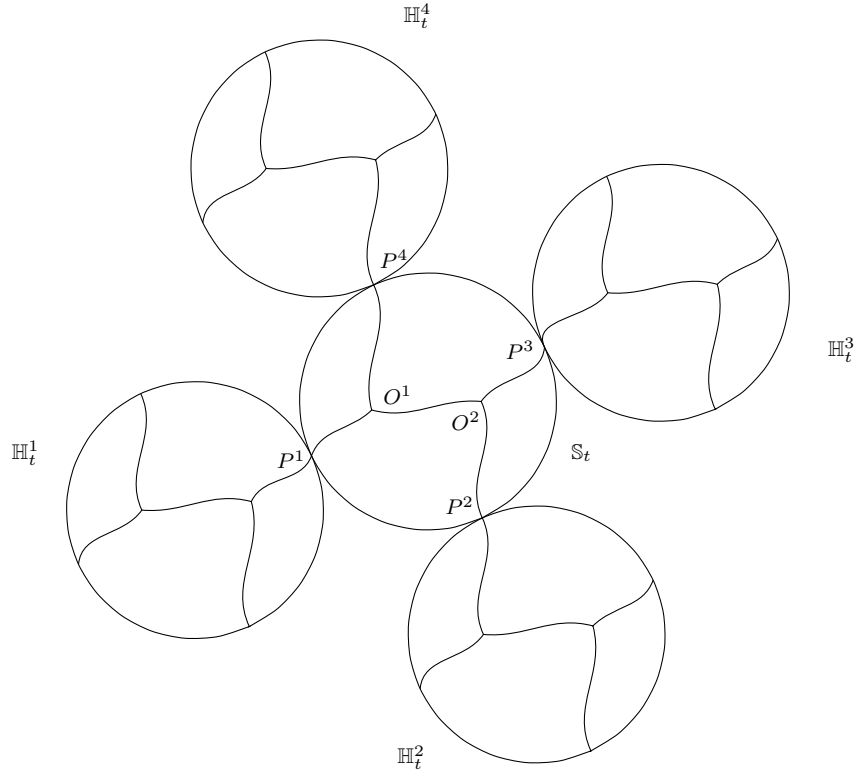


Figure 34: A tree-like network  $\mathbb{S}_t$  with the associated networks  $\mathbb{H}_t^i$ .

We define for the networks  $\mathbb{H}_t^i$  the functions  $E^i : [0, T) \rightarrow \mathbb{R}$ , analogous to the function  $E : [0, T) \rightarrow \mathbb{R}$  of  $\mathbb{S}_t$  and, for every  $t \in [0, T)$ , we call  $\Pi(t)$  the minimum of the values  $E^i(t)$ . The function  $\Pi : [0, T) \rightarrow \mathbb{R}$  is still a locally Lipschitz function (hence, differentiable for almost every time), clearly satisfying  $\Pi(t) \leq E^i(t) \leq E(t)$  for all  $t \in [0, T)$ . Moreover, as there are no self-intersections, by construction, we have  $\Pi(0) > 0$ . If we prove that  $\Pi(t) \geq C > 0$  for all  $t \in [0, T)$ , from some constant  $C \in \mathbb{R}$ , then, we can conclude that also  $E(t) \geq C > 0$ , for all  $t \in [0, T)$ .

**Theorem 14.5.** *Let  $\Omega$  be a open, bounded, strictly convex subset of  $\mathbb{R}^2$ . Let  $\mathbb{S}_0$  be an initial regular network with at most two triple junctions and let the  $\mathbb{S}_t$  be a smooth evolution by curvature of  $\mathbb{S}_0$ , defined in a maximal time interval  $[0, T)$ .*

*Then, there exists a constant  $C > 0$  depending only on  $\mathbb{S}_0$  such that  $E(t) \geq C > 0$ , for every  $t \in [0, T)$ . In particular, the networks  $\mathbb{S}_t$  remain (uniformly, in a sense) embedded during the flow.*

To prove this theorem we first show the next proposition and lemma.

**Proposition 14.6.** *Let  $t \in [0, T)$  such that*

- $0 < E(t) < 1/4$ ,

- for at least one minimizing pair  $(p, q)$  of  $\Phi_t$ , the curve  $\Gamma_{p,q}$  contains at most two triple junctions with neither  $p$  nor  $q$  coinciding with one of the end-points  $P^i$ .

Then, if the derivative  $\frac{dE(t)}{dt}$  exists, it is positive.

*Proof.* By simplicity, we consider in detail only the case shown in Figure 35. The computations in the other situations are analogous.

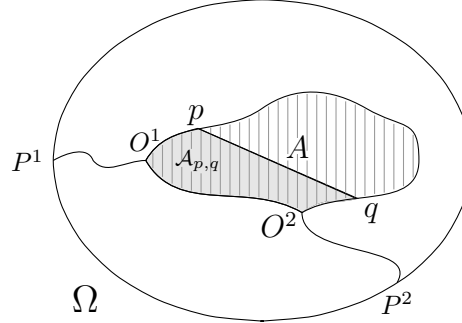


Figure 35: The situation considered in the computations of Proposition 14.6.

Let  $0 < E(t) < 1/4$  and let  $(p, q)$  a minimizing pair for  $\Phi_t$  such that the two points are both distinct from the end-points  $P^i$ . We choose a value  $\varepsilon > 0$  smaller than the “geodesic” distances of  $p$  and  $q$  from the 3-points of  $\mathbb{S}_t$  and between them.

Possibly taking a smaller  $\varepsilon > 0$ , we fix an arclength coordinate  $s \in (-\varepsilon, \varepsilon)$  and a local parametrization  $p(s)$  of the curve containing  $p$  such that  $p(0) = p$ , with the same orientation of the original one. Let  $\eta(s) = |p(s) - q|$ , since

$$E(t) = \min_{s \in (-\varepsilon, \varepsilon)} \frac{\eta^2(s)}{\psi(A_{p(s),q})} = \frac{\eta^2(0)}{\psi(A_{p,q})},$$

if we differentiate in  $s$  we obtain

$$\frac{d\eta^2(0)}{ds} \psi(A_{p(0),q}) = \frac{d\psi(A_{p(0),q})}{ds} \eta^2(0). \quad (14.1)$$

We underline that we are considering the function  $\psi$  because we are doing all the computation for the case shown in Figure 35, where there is a loop. For a network without loops the computations are simpler: instead of formula (14.1), one has

$$\frac{d\eta^2(0)}{ds} A_{p(0),q} = \frac{dA_{p(0),q}}{ds} \eta^2(0),$$

see [63, Page 281], for instance.

As the intersection of the segment  $\overline{pq}$  with the network is transversal, we have an angle  $\alpha(p) \in (0, \pi)$  determined by the unit tangent  $\tau(p)$  and the vector  $q - p$ .

We compute

$$\begin{aligned} \left. \frac{d\eta^2(s)}{ds} \right|_{s=0} &= -2 \langle \tau(p) | q - p \rangle = -2|p - q| \cos \alpha(p) \\ \left. \frac{dA(s)}{ds} \right|_{s=0} &= 0 \\ \left. \frac{dA_{p(s),q}}{ds} \right|_{s=0} &= \frac{1}{2} |\tau(p) \wedge (q - p)| = \frac{1}{2} \langle \nu(p) | q - p \rangle = \frac{1}{2} |p - q| \sin \alpha(p) \\ \left. \frac{d\psi(A_{p(s),q})}{ds} \right|_{s=0} &= \frac{dA_{p,q}}{ds} \cos \left( \frac{\pi}{A} A_{p,q} \right) \\ &= \frac{1}{2} |p - q| \sin \alpha(p) \cos \left( \frac{\pi}{A} A_{p,q} \right). \end{aligned}$$

Putting these derivatives in equation (14.1) and recalling that  $\eta^2(0)/\psi(A_{p,q}) = E(t)$ , we get

$$\cot \alpha(p) = -\frac{|p-q|^2}{4\psi(A_{p,q})} \cos\left(\frac{\pi}{A}A_{p,q}\right) = -\frac{E(t)}{4} \cos\left(\frac{\pi}{A}A_{p,q}\right). \quad (14.2)$$

Since  $0 < E(t) < \frac{1}{4} < 4(2 - \sqrt{3})$ , we have  $\sqrt{3} - 2 < \cot \alpha(p) < 0$ , which implies

$$\frac{\pi}{2} < \alpha(p) < \frac{7\pi}{12}. \quad (14.3)$$

The same argument clearly holds for the point  $q$ , hence defining  $\alpha(q) \in (0, \pi)$  to be the angle determined by the unit tangent  $\tau(q)$  and the vector  $p-q$ , by equation (14.2) it follows that  $\alpha(p) = \alpha(q)$  and we simply write  $\alpha$  for both.

We consider now a different variation, moving at the same time the points  $p$  and  $q$ , in such a way that  $\frac{dp(s)}{ds} = \tau(p(s))$  and  $\frac{dq(s)}{ds} = \tau(q(s))$ .

As above, letting  $\eta(s) = |p(s) - q(s)|$ , by minimality we have

$$\begin{aligned} \frac{d\eta^2(0)}{ds} \psi(A_{p(s),q(s)}) \Big|_{s=0} &= \left( \frac{d\psi(A_{p(s),q(s)})}{ds} \Big|_{s=0} \right) \eta^2(0) \quad \text{and} \\ \frac{d^2\eta^2(0)}{ds^2} \psi(A_{p(s),q(s)}) \Big|_{s=0} &\geq \left( \frac{d^2\psi(A_{p(s),q(s)})}{ds^2} \Big|_{s=0} \right) \eta^2(0). \end{aligned} \quad (14.4)$$

Computing as before,

$$\begin{aligned} \frac{d\eta^2(s)}{ds} \Big|_{s=0} &= 2\langle p-q \mid \tau(p) - \tau(q) \rangle = -4|p-q| \cos \alpha \\ \frac{dA_{p(s),q(s)}}{ds} \Big|_{s=0} &= -\frac{1}{2}\langle p-q \mid \nu(p) + \nu(q) \rangle = +|p-q| \sin \alpha \\ \frac{d^2\eta^2(s)}{ds^2} \Big|_{s=0} &= 2\langle \tau(p) - \tau(q) \mid \tau(p) - \tau(q) \rangle + 2\langle p-q \mid k(p)\nu(p) - k(q)\nu(q) \rangle \\ &= 2|\tau(p) - \tau(q)|^2 + 2\langle p-q \mid k(p)\nu(p) - k(q)\nu(q) \rangle \\ &= 8\cos^2 \alpha + 2\langle p-q \mid k(p)\nu(p) - k(q)\nu(q) \rangle \\ \frac{d^2A_{p(s),q(s)}}{ds^2} \Big|_{s=0} &= -\frac{1}{2}\langle \tau(p) - \tau(q) \mid \nu(p) + \nu(q) \rangle + \frac{1}{2}\langle p-q \mid k(p)\tau(p) + k(q)\tau(q) \rangle \\ &= -\frac{1}{2}\langle \tau(p) \mid \nu(q) \rangle + \frac{1}{2}\langle \tau(q) \mid \nu(p) \rangle \\ &\quad + \frac{1}{2}\langle p-q \mid k(p)\tau(p) + k(q)\tau(q) \rangle \\ &= -2\sin \alpha \cos \alpha - 1/2|p-q|(k(p) - k(q)) \cos \alpha \\ \frac{d^2\psi(A_{p(s),q(s)})}{ds^2} \Big|_{s=0} &= \frac{d}{ds} \left\{ \frac{dA_{p(s),q(s)}}{ds} \cos\left(\frac{\pi}{A}A_{p(s),q(s)}\right) \right\} \Big|_{s=0} \\ &= (-2\sin \alpha \cos \alpha - \frac{1}{2}|p-q|(k(p) - k(q)) \cos \alpha) \cos\left(\frac{\pi}{A}A_{p,q}\right) \\ &\quad - \frac{\pi}{A}|p-q|^2 \sin^2 \alpha \sin\left(\frac{\pi}{A}A_{p,q}\right). \end{aligned}$$

Substituting the last two relations in inequality (14.4), we get

$$\begin{aligned} &(8\cos^2 \alpha + 2\langle p-q \mid k(p)\nu(p) - k(q)\nu(q) \rangle)\psi(A_{p,q}) \\ &\geq |p-q|^2 \left\{ (-2\sin \alpha \cos \alpha - \frac{1}{2}|p-q|(k(p) - k(q)) \cos \alpha) \cos\left(\frac{\pi}{A}A_{p,q}\right) \right. \\ &\quad \left. - \frac{\pi}{A}|p-q|^2 \sin^2 \alpha \sin\left(\frac{\pi}{A}A_{p,q}\right) \right\}, \end{aligned}$$

hence, keeping in mind that  $\tan \alpha = \frac{-4}{E(t) \cos\left(\frac{\pi}{A} A_{p(s), q(s)}\right)}$ , we obtain

$$\begin{aligned}
& 2\psi(A_{p,q}) \langle p - q \mid k(p)\nu(p) - k(q)\nu(q) \rangle + 1/2 |p - q|^3 (k(p) - k(q)) \cos \alpha \cos \left( \frac{\pi}{A} A_{p,q} \right) \\
& \geq -2 \sin \alpha \cos \alpha |p - q|^2 \cos \left( \frac{\pi}{A} A_{p,q} \right) \\
& \quad - 8\psi(A_{p,q}) \cos^2 \alpha + |p - q|^4 \sin^2 \alpha \left[ -\frac{\pi}{A} \sin \left( \frac{\pi}{A} A_{p,q} \right) \right] \\
& = -2\psi(A_{p,q}) \cos^2 \alpha \left( \tan \alpha \frac{|p - q|^2}{\psi(A_{p,q})} \cos \left( \frac{\pi}{A} A_{p,q} \right) + 4 \right) \\
& \quad + |p - q|^4 \sin^2 \alpha \left[ -\frac{\pi}{A} \sin \left( \frac{\pi}{A} A_{p,q} \right) \right] \\
& = |p - q|^4 \sin^2 \alpha \left[ -\frac{\pi}{A} \sin \left( \frac{\pi}{A} A_{p,q} \right) \right]. \tag{14.5}
\end{aligned}$$

We now compute the derivative  $\frac{dE(t)}{dt}$  by means of the Hamilton's trick (see [37] or [61, Lemma 2.1.3]), that is,

$$\frac{dE(t)}{dt} = \frac{\partial}{\partial t} \Phi_t(\bar{p}, \bar{q}),$$

for *any* minimizing pair  $(\bar{p}, \bar{q})$  for  $\Phi_t$ . In particular,  $\frac{dE(t)}{dt} = \frac{\partial}{\partial t} \Phi_t(p, q)$  and, we recall,  $\frac{|p - q|^2}{\psi(A_{p,q})} = E(t)$ . Notice that by minimality of the pair  $(p, q)$ , we are free to choose the “motion” of the points  $p(s)$ ,  $q(s)$  “inside” the networks  $\Gamma_s$  in computing such partial derivative, that is,

$$\frac{dE(t)}{dt} = \frac{\partial}{\partial t} \Phi_t(p, q) = \frac{d}{ds} \Phi_t(p(s), q(s)) \Big|_{s=t}.$$

Since locally the networks are moving by curvature and we know that neither  $p$  nor  $q$  coincides with the 3-point, we can find  $\varepsilon > 0$  and two smooth curves  $p(s), q(s) \in \Gamma_s$  for every  $s \in (t - \varepsilon, t + \varepsilon)$  such that

$$\begin{aligned}
p(t) = p \quad & \text{and} \quad \frac{dp(s)}{ds} = k(p(s), s) \nu(p(s), s), \\
q(t) = q \quad & \text{and} \quad \frac{dq(s)}{ds} = k(q(s), s) \nu(q(s), s).
\end{aligned}$$

Then,

$$\frac{dE(t)}{dt} = \frac{\partial}{\partial t} \Phi_t(p, q) = \frac{1}{[\psi(A_{p,q})]^2} \left( \psi(A_{p,q}) \frac{d|p(s) - q(s)|^2}{ds} - |p - q|^2 \frac{d\psi(A_{p(s), q(s)})}{ds} \right) \Big|_{s=t}. \tag{14.6}$$

With a straightforward computation, we get the following equalities,

$$\begin{aligned}
\frac{d|p(s) - q(s)|^2}{ds} \Big|_{s=t} &= 2 \langle p - q \mid k(p)\nu(p) - k(q)\nu(q) \rangle \\
\frac{dA(s)}{ds} \Big|_{s=t} &= -\frac{4\pi}{3} \\
\frac{dA_{p(s), q(s)}}{ds} \Big|_{s=t} &= \int_{\Gamma_{p,q}} \langle \underline{k}(s) \mid \nu_{\xi_{p,q}} \rangle ds + \frac{1}{2} |p - q| \langle \nu_{[p,q]} \mid k(p)\nu(p) + k(q)\nu(q) \rangle \\
&= 2\alpha - \frac{4\pi}{3} - \frac{1}{2} |p - q| (k(p) - k(q)) \cos \alpha \\
\frac{d\psi(A_{p(s), q(s)})}{ds} \Big|_{s=t} &= -\frac{4\pi}{3} \left[ \frac{1}{\pi} \sin \left( \frac{\pi}{A} A_{p,q} \right) - \frac{A_{p,q}}{A} \cos \left( \frac{\pi}{A} A_{p,q} \right) \right] \\
&\quad + \left( 2\alpha - \frac{4\pi}{3} - \frac{1}{2} |p - q| (k(p) - k(q)) \cos \alpha \right) \cos \left( \frac{\pi}{A} A_{p,q} \right)
\end{aligned}$$

where we wrote  $\nu_{\xi_{p,q}}$  and  $\nu_{[p,q]}$  for the exterior unit normal vectors to the region  $A_{p,q}$ , respectively at the points of the geodesic  $\xi_{p,q}$  and of the segment  $\overline{pq}$ .

We remind that in general  $\frac{dA(t)}{dt} = -(2 - m/3)\pi$  where  $m$  is the number of triple junctions of the loop (see formula (8.2)), hence, we have  $\frac{dA(t)}{dt} = -\frac{4\pi}{3}$ , since we are referring to the situation in Figure 35, where there is a loop with exactly two triple junctions.

Substituting these derivatives in equation (14.6) we get

$$\begin{aligned} \frac{dE(t)}{dt} &= \frac{2\langle p - q | k(p)\nu(p) - k(q)\nu(q) \rangle}{\psi(A_{p,q})} \\ &\quad - \frac{|p - q|^2}{[\psi(A_{p,q})]^2} \left\{ -\frac{4\pi}{3} \left[ \frac{1}{\pi} \sin\left(\frac{\pi}{A} A_{p,q}\right) - \frac{A_{p,q}}{A} \cos\left(\frac{\pi}{A} A_{p,q}\right) \right] \right. \\ &\quad \left. + \left( 2\alpha - \frac{4\pi}{3} - \frac{1}{2}|p - q|(k(p) - k(q)) \cos \alpha \right) \cos\left(\frac{\pi}{A} A_{p,q}\right) \right\} \end{aligned}$$

and, by equation (14.5),

$$\begin{aligned} \frac{dE(t)}{dt} &\geq -\frac{|p - q|^2}{[\psi(A_{p,q})]^2} \left\{ -\frac{4\pi}{3} \sin\left(\frac{\pi}{A} A_{p,q}\right) + \frac{4\pi}{3} \frac{A_{p,q}}{A} \cos\left(\frac{\pi}{A} A_{p,q}\right) \right. \\ &\quad \left. + \left( 2\alpha - \frac{4\pi}{3} \right) \cos\left(\frac{\pi}{A} A_{p,q}\right) + \frac{\pi}{A} |p - q|^2 \sin^2(\alpha) \sin\left(\frac{\pi}{A} A_{p,q}\right) \right\}. \end{aligned}$$

It remains to prove that the quantity

$$\begin{aligned} &\frac{4\pi}{3} \sin\left(\frac{\pi}{A} A_{p,q}\right) - \frac{4\pi}{3} \frac{A_{p,q}}{A} \cos\left(\frac{\pi}{A} A_{p,q}\right) + \left( \frac{4\pi}{3} - 2\alpha \right) \cos\left(\frac{\pi}{A} A_{p,q}\right) \\ &- \frac{\pi}{A} |p - q|^2 \sin^2(\alpha) \sin\left(\frac{\pi}{A} A_{p,q}\right) \end{aligned}$$

is positive.

As  $E(t) = \frac{|p-q|^2}{\psi(A_{p,q})} = \frac{|p-q|^2}{\frac{4\pi}{3} \sin(\frac{\pi}{A} A_{p,q})}$ , we can write

$$\begin{aligned} &\frac{4\pi}{3} \sin\left(\frac{\pi}{A} A_{p,q}\right) - \frac{4\pi}{3} \frac{A_{p,q}}{A} \cos\left(\frac{\pi}{A} A_{p,q}\right) + \left( \frac{4\pi}{3} - 2\alpha \right) \cos\left(\frac{\pi}{A} A_{p,q}\right) \\ &- \frac{\pi}{A} |p - q|^2 \sin^2(\alpha) \sin\left(\frac{\pi}{A} A_{p,q}\right) \\ &= \frac{4\pi}{3} \sin\left(\frac{\pi}{A} A_{p,q}\right) - \frac{4\pi}{3} \frac{A_{p,q}}{A} \cos\left(\frac{\pi}{A} A_{p,q}\right) + \left( \frac{4\pi}{3} - 2\alpha \right) \cos\left(\frac{\pi}{A} A_{p,q}\right) \\ &- E(t) \sin^2(\alpha) \sin^2\left(\frac{\pi}{A} A_{p,q}\right). \end{aligned}$$

Notice that using inequality (14.3), we can evaluate  $\frac{4\pi}{3} - 2\alpha \in (\pi/6, \pi/3)$ , in particular, it is positive.

We finally conclude the estimate of  $\frac{dE(t)}{dt}$  and the proof of this proposition by separating the analysis in two cases, depending on the value of  $\frac{A_{p,q}}{A}$ .

If  $0 \leq \frac{A_{p,q}}{A} \leq \frac{1}{3}$ , we have

$$\begin{aligned} \frac{dE(t)}{dt} &\geq \frac{4\pi}{3} \sin\left(\frac{\pi}{A} A_{p,q}\right) - \frac{4\pi}{3} \frac{A_{p,q}}{A} \cos\left(\frac{\pi}{A} A_{p,q}\right) \\ &\quad + \left( \frac{4\pi}{3} - 2\alpha \right) \cos\left(\frac{\pi}{A} A_{p,q}\right) - E(t) \sin^2(\alpha) \sin^2\left(\frac{\pi}{A} A_{p,q}\right) \\ &\geq \left( \frac{4\pi}{3} - 2\alpha \right) \cos\left(\frac{\pi}{A} A_{p,q}\right) - E(t) \sin^2(\alpha) \sin^2\left(\frac{\pi}{A} A_{p,q}\right) \\ &\geq \left( \frac{\pi}{6} \right) \cos\left(\frac{\pi}{3}\right) - E(t) \sin^2\left(\frac{\pi}{3}\right) > 0. \end{aligned}$$



If  $\frac{1}{3} \leq \frac{A_{p,q}}{A} \leq \frac{1}{2}$ , we get

$$\begin{aligned}
\frac{dE(t)}{dt} &\geq \frac{4}{3} \sin\left(\frac{\pi}{A} A_{p,q}\right) - \frac{4\pi}{3} \frac{A_{p,q}}{A} \cos\left(\frac{\pi}{A} A_{p,q}\right) \\
&\quad + \left(\frac{4\pi}{3} - 2\alpha\right) \cos\left(\frac{\pi}{A} A_{p,q}\right) - E(t) \sin^2(\alpha) \sin^2\left(\frac{\pi}{A} A_{p,q}\right) \\
&\geq \frac{4}{3} \sin\left(\frac{\pi}{A} A_{p,q}\right) - \frac{4\pi}{3} \frac{A_{p,q}}{A} \cos\left(\frac{\pi}{A} A_{p,q}\right) - E(t) \sin^2(\alpha) \sin^2\left(\frac{\pi}{A} A_{p,q}\right) \\
&\geq \frac{4}{3} \left(\sin\left(\frac{\pi}{3}\right) - \frac{\pi}{3} \cos\left(\frac{\pi}{3}\right)\right) - E(t) > 0.
\end{aligned}$$

□

*Remark 14.7.* We want to stress here the reason why we are able to prove Proposition 14.6 only when  $\Gamma_{p,q}$  contains at most two triple junctions and so Theorem 14.5 only for networks with at most two 3-points. If we try to repeat the computations of the final part of this proof considering a situation such that  $\Gamma_{p,q}$  contains more than two triple junctions, as the value of  $\frac{dA(t)}{dt}$  changes according to  $\frac{dA(t)}{dt} = -(2-m/3)\pi$ , when  $m \geq 3$ , we only have  $\frac{dA(t)}{dt} \geq -\pi$  (instead of being equal to  $-4\pi/3$ ), which is not sufficient to get to the inequality  $\frac{dE(t)}{dt} > 0$ .

**Lemma 14.8.** *Let  $\Omega$  be a open, bounded, strictly convex subset of  $\mathbb{R}^2$ . Let  $\mathbb{S}_0$  be an initial regular network with two triple junctions and let the  $\mathbb{S}_t$  be the evolution by curvature of  $\mathbb{S}_0$  defined in a maximal time interval  $[0, T)$ . Then, there cannot be a sequence of times  $t_j \rightarrow T$  such that, along such sequence, the two triple junctions converge to the same end-point of the network.*

*Proof.* Let  $O^1(t)$  and  $O^2(t)$  be the two triple junctions of  $\mathbb{S}_t$  and  $P^i$  the end-points on  $\partial\Omega$ . Suppose, by contradiction, that  $\lim_{i \rightarrow \infty} O^j(t_i) = P^1$ , for  $j \in \{1, 2\}$ . Notice that if  $\mathbb{S}_t$  is not a tree, then it has the structure either of a “lens/fish-shaped” network (see Figure 7) or of an “island-shaped” network.

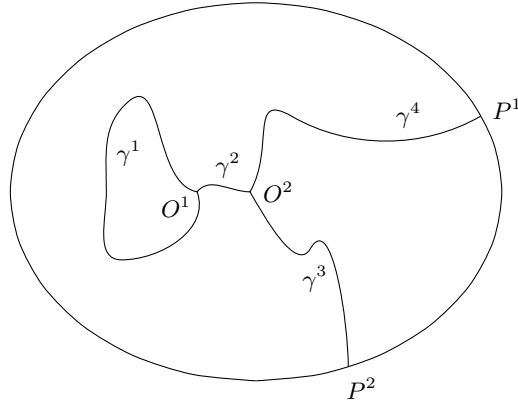


Figure 36: An island-shaped network.

If we consider the sequence of rescaled networks  $\tilde{\mathbb{H}}_{P^1, t_j}^1$  obtained via Huisken’s dynamical procedure applied to  $\mathbb{H}_t^1$ , as in Proposition 8.20, centered in  $P^1$ , it converges in  $C_{\text{loc}}^{1,\alpha} \cap W_{\text{loc}}^{2,2}$ , for any  $\alpha \in (0, 1/2)$  to a (not empty) limit degenerate regular shrinker  $\tilde{\mathbb{H}}_\infty$ . We analyze the possible  $\tilde{\mathbb{H}}_\infty$  without using the multiplicity-one conjecture **M1**, to avoid a “circular argument”. If the blow up limit  $\tilde{\mathbb{H}}_\infty$  is a line for the origin or four halflines forming angles of  $120/60$  degrees, then, in both cases the curvature of the non-rescaled networks  $\mathbb{H}_t$  (hence, of  $\mathbb{S}_t$ ) is locally uniformly bounded around  $P^1$  (by White’s regularity theorem in [84] and Proposition 10.21, which are both independent of **M1**) and they (in the second case, by arguing as in Lemma 8.18) “forbid” the presence of another 3-point of  $\mathbb{S}_t$  in a space-time neighborhood of  $(P^1, T)$ , clearly contradicting the hypotheses.

If  $\mathbb{S}_t$  contains a loop, it cannot vanish in the rescaling procedure, going to infinity (its curves would

converge to two distinct halflines) or collapsing to a core (for instance in the origin), since the area of the region bounded by the rescaled loop in  $\tilde{\mathbb{S}}_{P^1, t}$  is constant in  $t$  (see Section 8.2). Hence, such loop would be present in  $\tilde{\mathbb{H}}_\infty$  and actually could only have the structure of a “Brakke spoon” (see Figure 6) or of a “lens/fish-shaped” network (see Figure 8). It is then easy to see that, being part of a shrinker, it must contain the origin of  $\mathbb{R}^2$  in its inside, which is clearly not possible in our situation. The last case we have to deal with is when  $\mathbb{S}_t$  is a tree. It follows that also  $\tilde{\mathbb{S}}_\infty$  and so  $\tilde{\mathbb{H}}_\infty$  are trees and the same for the underlying graph is tree, then  $\tilde{\mathbb{H}}_\infty$  is a symmetric family of halflines from the origin, by Lemma 8.9. The cases when  $\tilde{\mathbb{H}}_\infty$  has no core or the core is a single segment are the ones described above (we concluded that the other 3-points cannot tend to  $P^1$ , a contradiction). The remaining case is when the (symmetric) core of  $\tilde{\mathbb{H}}_\infty$  is given by three degenerate curves (and four 3-points) at the origin. In this case it is straightforward to see that  $\tilde{\mathbb{S}}_\infty$  contains a straight line for the origin, which is not possible since  $\tilde{\mathbb{S}}_\infty$  must be contained in an angle with opening less than  $\pi$ , by the strict convexity of  $\Omega$ , as it is shown in Proposition 8.12.  $\square$

*Remark 14.9.* As before, we remark that the strictly convexity hypothesis on  $\Omega$  can actually be weakened asking that  $\Omega$  is convex and that there does not exist three aligned end-points of the initial network  $\mathbb{S}_0$  on  $\partial\Omega$ .

*Proof of Theorem 14.5.* If  $\mathbb{S}_t$  is the evolution of a network with only one triple junction, any of the evolving networks  $\mathbb{H}_t^i$  has exactly two 3-points. Let  $t \in [0, T)$  a time such that  $0 < \Pi(t) < 1/4$  and  $\Pi$  and all embeddedness measures  $E^i$ , associated to the networks  $\mathbb{H}_t^i$ , are differentiable at  $t$  (this clearly holds for almost every time).

Let  $E^i(t) = \Pi(t) < 1/4$  and  $E^i(t)$  is realized by a pair of points  $p$  and  $q$  in  $\mathbb{H}_t^i$ , we separate the analysis in the following cases:

- If the points  $p$  and  $q$  of the minimizing pair are both end-points of  $\mathbb{H}_t^i$ , by construction  $|p - q| \geq \varepsilon > 0$ . Moreover, the area enclosed in the Jordan curve formed by the segment  $\overline{pq}$  and by the geodesic curve  $\Gamma_{p,q}$  can be uniformly bounded by above by a constant  $\overline{C} > 0$ , for instance, the area of a ball containing all the networks  $\mathbb{H}_t^i$ . Since  $\varepsilon > 0$  and  $\overline{C}$  depend only on  $\Omega$  and on the structure of the initial network  $\mathbb{S}_0$  (more precisely on the position of the end-points on the boundary of  $\Omega$ , that stay fixed during the evolution and that do not coincide), the ratio  $\frac{|p-q|^2}{\psi(A_{p,q})}$  (or  $\frac{|p-q|^2}{A_{p,q}}$ , if  $p, q$  do not belong to a loop) is greater or equal than some constant  $C_\varepsilon = \frac{\varepsilon^2}{\overline{C}} > 0$  uniformly, hence the same holds for  $\Pi(t)$ .
- If one point is internal and the other is an end-point of  $\mathbb{H}_t^i$ , we consider the following two situations. If one of the two point  $p$  and  $q$  is in  $\mathbb{S}_t \subset \mathbb{H}_t^i$  and the other is in the reflected network  $\mathbb{S}_t^{R_i}$ , then, we obtain, by construction, a uniform bound from below on  $\Pi(t)$  as in the case in which  $p$  and  $q$  are both boundary points of  $\mathbb{H}_t^i$ . Otherwise, if  $p$  and  $q$  are both in  $\mathbb{S}_t$  and one of them coincides with  $P^j$  with  $j \neq i$ , either the other point coincides with  $P^i$  and we have again a uniform bound from below on  $\Pi(t)$ , as before, or both  $p$  and  $q$  are points of  $\mathbb{H}_t^j$  both not coinciding with its end-points and  $E^j(t) = E^i(t) = \Pi(t) < 1/4$ , so we can apply the argument at the next point.
- If  $p$  and  $q$  are both “inside”  $\mathbb{H}_t^i$ , by Hamilton’s trick (see [37] or [61, Lemma 2.1.3]), we have  $\frac{d\Pi(t)}{dt} = \frac{dE^i(t)}{dt}$  and, by Proposition 14.6,  $\frac{dE^i(t)}{dt} > 0$ , hence  $\frac{d\Pi(t)}{dt} > 0$ .

All this discussion implies that at almost every point  $t \in [0, T)$  such that  $\Pi(t)$  is smaller than some uniform constant depending only on  $\Omega$  and on the structure of the initial network  $\mathbb{S}_0$ , then  $\frac{d\Pi(t)}{dt} > 0$ , which clearly proves the theorem in the case a network with a single triple junction (see also [63, Section 4]).

Let now  $\mathbb{S}_t$  be a flow of regular networks with *two* triple junctions. If there are no end-points, the conclusion follows immediately from Proposition 14.6. Hence, we assume that  $\mathbb{S}_t$  has two or four end-points (in the first case there is a loop, in the second  $\mathbb{S}_t$  is a tree), which are the only possibilities. The analysis is the same as above, with only a delicate point to be addressed, that is, in the last case, when the two points  $p$  and  $q$  of the minimizing pair are “inside”  $\mathbb{H}_t^i$  and we apply Proposition 14.6.

Indeed, since  $\mathbb{H}_t^i$  has *four* 3-points it can happen that the geodesic curve  $\Gamma_{p,q}$  contains more than two 3-points, hence this case requires a special treatment. Notice that if the points  $p$  and  $q$  are both “inside”  $\mathbb{S}_t \subset \mathbb{H}_t^i$ , then Proposition 14.6 applies and we are done. We then assume that  $p \in \mathbb{S}_t$ ,  $q \in \mathbb{S}_t^{R_i}$ , and  $\Gamma_{p,q}$  contains more than two triple junctions.

We want to show that there exists a uniform positive constant  $\varepsilon$  such that  $|p - q| \geq \varepsilon > 0$ , which implies a uniform positive estimate from below on  $E^i(t)$ , as above. This will conclude the proof.

Assume by contradiction that such a bound is not possible, then, for a sequence of times  $t_j \rightarrow T$ , the Euclidean distance between the two points  $p_j$  and  $q_j$  of the associated minimizing pair of  $\Phi_{t_j}$  goes to zero, as  $j \rightarrow \infty$ , and this can happen only if  $p_i, q_i \rightarrow P^i$ . It follows, by the maximum principle that the two 3-points  $O^1(t)$  and  $O^2(t)$  converge to  $P^i$  on some sequence of times  $t_k \rightarrow T$  (possibly different from  $t_j$ ), which is forbidden by Lemma 14.8 and we are done.  $\square$

*Remark 14.10.* Notice, by inspecting the previous proof, that in the case that  $\mathbb{S}_t$  has a single 3-point, the strict convexity of  $\Omega$  is not necessary, convexity is sufficient.

## 14.1 Consequences for the multiplicity-one conjecture

The quantity  $E(t)$  considered in the previous section is clearly, by definition, dilation and translation invariant, moreover it is continuous under  $C_{\text{loc}}^1$ -convergence of networks. Hence, if  $E(t) \geq C > 0$  for every  $t \in [0, T)$ , the same holds for every  $C_{\text{loc}}^1$ -limit of rescalings of networks of the flow  $\mathbb{S}_t$ . This clearly implies the strong multiplicity-one conjecture **SM1**.

**Corollary 14.11.** *If  $\Omega$  is strictly convex and the initial network  $\mathbb{S}_0$  has at most two triple junctions, then the strong multiplicity-one conjecture **SM1** is true for the flow  $\mathbb{S}_t$ .*

A by-product of the proofs of Proposition 14.6 and Theorem 14.5 is actually that also the function  $\Pi(t)$  is positively uniformly bounded from below during the flow.

**Corollary 14.12.** *If  $\Omega$  is strictly convex and the initial network  $\mathbb{S}_0$  has at most two triple junctions, then the strong multiplicity-one conjecture **SM1** is true for all the “symmetrized” flows  $\mathbb{H}_t^i$ .*

*Remark 14.13.* Actually, in general, if we are able to show the (strong) multiplicity-one conjecture for a curvature flow  $\mathbb{S}_t$  in a strictly convex open set  $\Omega$ , then, by construction and Proposition 8.12, it also holds for all the “symmetrized” flows  $\mathbb{H}_t^i$ . This remark is in order since in the analysis of the flow  $\mathbb{S}_t$  in the previous sections, we used the “reflection” argument at the end-points of the network  $\mathbb{S}_t$ , then we argued applying **M1** to the resulting networks  $\mathbb{H}_t^i$  (to be precise, in Section 10.1 and in the proofs of Proposition 10.14 and of Proposition 10.20).

Another situation that can be analyzed by means of the ideas of this section is the following.

**Proposition 14.14.** *If during the curvature flow of a tree  $\mathbb{S}_t$  the triple junctions stay uniformly far from each other and from the end-points, then **SM1** is true for the flows  $\mathbb{S}_t$  and all  $\mathbb{H}_t^i$ . As a consequence, the evolution of  $\mathbb{S}_t$  does not develop singularities at all.*

*Proof.* We divide all the pairs of curves of the evolving tree  $\mathbb{S}_t$  in two families, depending if the curve of a pair have a common 3-point or not. In the second case, by means of maximum principle and the assumption on the 3-points, there is a uniform constant  $C > 0$  such that any couple of points, one on each curve of such pair, have distance bounded below by  $C$ . Then, if the pair of points of  $\mathbb{S}_t$  realizing the quantity  $E(t)$  stay on such curves it follows  $E(t) \geq C' > 0$  for some uniform constant  $C'$ . In case  $E(t) < C'$ , it follows that such pair of points either stay on the same curve or on two curves with a common 3-point. Hence, the “geodesic” curve  $\Gamma_{p,q}$  contains at most one 3-point, being  $\mathbb{S}_t$  a tree. This implies that  $\frac{dE(t)}{dt} > 0$ , by Proposition 14.3. Then, the strong multiplicity-one conjecture follows for  $\mathbb{S}_t$  and for all the “symmetrized” flows  $\mathbb{H}_t^i$ , by the same argument in the proof of Theorem 14.5, taking into account the hypothesis that the triple junctions stay uniformly far also from the end-points.

Then, the only possible singularities of the flow are given by the collapse of a curve of the network, but this is excluded by the hypotheses, hence the flow is smooth for all times.  $\square$

## 15 The flow of networks with only one triple junction

If we consider the possible (topological) structures of regular networks with only one triple junction, we see that there are only two possible cases: a *triod*  $\mathbb{T}$  or a *spoon*  $\Gamma$ . As the triod is simplest configuration of an “essentially” singular one-dimensional set to let evolve by curvature, a spoon is the simplest case with a loop.

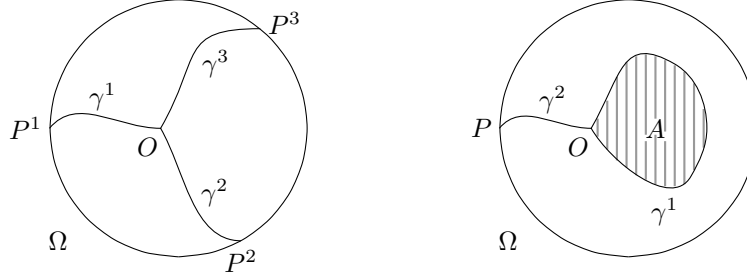


Figure 37: Networks with only one triple junction: the triod and the “spoon” network.

As an example of the analysis of the previous sections, we can completely describe the evolution of these two networks (see [59, 63, 70]) in a strictly convex, open and smooth subset  $\Omega \subset \mathbb{R}^2$ , since the strong multiplicity-one conjecture **SM1** holds in these cases (see Section 14.1).

As we will see in Theorem 15.2 in the case of the triod we can exclude the presence of singularities till the lengths of the three curves stay positively bounded below, in the case of the spoon instead a singularity eventually arises.

As defined in Section 4, fixed a smooth, open, strictly convex set  $\Omega \subset \mathbb{R}^2$ , a triod is a network (a tree)  $\mathbb{T}$  composed only of three regular, embedded  $C^1$  curves  $\gamma^i : [0, 1] \rightarrow \bar{\Omega}$ . These curves intersect each other only at a single 3-point  $O$ , forming angles of 120 degrees, that is,  $\gamma^1(0) = \gamma^2(0) = \gamma^3(0) = O$  and have the other three end-points  $P^1, P^2, P^3$  on the boundary of  $\Omega$  with  $\gamma^i(1) = P^i$ , for  $i \in \{1, 2, 3\}$ .

A spoon  $\Gamma = \gamma^1([0, 1]) \cup \gamma^2([0, 1])$  is the union of two regular, embedded  $C^1$  curves  $\gamma^1, \gamma^2 : [0, 1] \rightarrow \bar{\Omega}$  which intersect each other only at a triple junction  $O$ , with angles of 120 degrees, that is,  $\gamma^1(0) = \gamma^1(1) = \gamma^2(0) = O \in \Omega$  and  $\gamma^2(1) = P \in \partial\Omega$ . We call  $\gamma^1$  the “closed” curve and  $\gamma^2$  the “open” curve of the spoon and we denote with  $A$  the area of the region enclosed in the loop given by  $\gamma^1$ .

For simplicity, we assume in the following that both the initial networks are smooth, hence Theorem 4.18 applies and gives a smooth curvature flow in a maximal time interval  $[0, T)$ .

As we discussed in the previous sections, in the more general case that the initial network is less regular (but the compatibility conditions, see Definitions 4.6 and 4.10, are satisfied), we need Theorems 4.8 and 4.13 to start the flow. If the initial network does not satisfy the compatibility conditions or it is not regular, we need to apply Theorem 6.8 or resort to Theorem 11.1 to have a curvature flow. Anyway, in all these cases, the flow is smooth for every positive time.

Regarding uniqueness (geometric uniqueness to be more precise, see Definition 4.21), instead, the fulfillment of the compatibility conditions is needed, see Proposition 4.22 Remarks 6.9 and 11.27.

We recall only the result for a smooth initial network (see 4.17 4.18 4.20 and 4.23).

**Proposition 15.1.** *Let  $\Omega \subset \mathbb{R}^2$  be a smooth, open, strictly convex set, then, for any smooth regular initial triod  $\mathbb{T}_0$  or any smooth regular initial spoon  $\Gamma_0$  in  $\Omega$ , there exists a geometrically unique smooth curvature flow in a maximal time interval  $[0, T)$ .*

Before proceeding, we also recall that during the flow the evolving networks stay embedded and intersect the boundary of  $\Omega$  only at the fixed end-points (transversally), see Section 8.2.

### 15.1 The triod

Suppose that  $T < +\infty$ , then, by Proposition 10.8 the lengths of the three curves cannot be uniformly positively bounded from below. Hence, as  $\Omega$  is strictly convex, Corollary 10.25 and Theorem 10.28

imply that the curvature of  $\mathbb{T}_t$  is uniformly bounded and there must be a “collapse” of a curve to a fixed end–point on  $\partial\Omega$ , when  $t \rightarrow T$ , as depicted in the right side of Figure 14.

Suppose instead that  $T = +\infty$ . Then, by Proposition 13.5, for every sequence of times  $t_i \rightarrow \infty$ , there exists a (not relabeled) subsequence such that the evolving triod  $\mathbb{T}_{t_i}$  converge in  $C^1$  to a possibly degenerate regular triod, embedded (by Theorem 14.5) and with zero curvature, as  $i \rightarrow \infty$ , that is, a Steiner triod connecting the three fixed points  $P^i$  on  $\partial\Omega$  (which possibly have a zero–length degenerate curve, for instance if the three end–points are the vertices of a triangle with an angle of 120 degrees). Moreover, as the Steiner triod connecting three points (and length minimizing) is unique, if it exists, for every subsequence of times we have the same limit triod, hence, the full sequence of triods  $\mathbb{T}_t$  converge to such limit, as  $t \rightarrow +\infty$ .

**Theorem 15.2.** *For any smooth, embedded, regular initial triod  $\mathbb{T}_0$  in a smooth, strictly convex open set  $\Omega \subset \mathbb{R}^2$ , with fixed end–points  $P^1, P^2, P^3 \in \partial\Omega$ , there exists a unique smooth evolution by curvature of  $\mathbb{T}_0$  which at every time is a smooth embedded regular triod in  $\Omega$ , in a maximal time interval  $[0, T)$ .*

*If  $T$  is finite, then a curve collapses to an end–point, when  $t \rightarrow T$  while the curvature remains bounded.*

*If  $T = +\infty$ , then the triods  $\mathbb{T}_t$  tend, as  $t \rightarrow +\infty$ , to the unique Steiner (length minimizing) embedded triod (possibly degenerate) connecting the three fixed end–points.*

We notice that there is an obvious example where the length of one curve goes to zero in finite time: consider an initial triod  $\mathbb{T}_0$  with the boundary points  $P^i$  on  $\partial\Omega$ , such that one angle of the triangle with vertices  $P^1, P^2, P^3$  is greater than 120 degrees. In this case the Steiner triod does not exist, hence the maximal time of a smooth evolution must be finite.

Instead, if the angles of the triangle with vertices  $P^1, P^2, P^3$  are all smaller than 120 degrees and the initial triod  $\mathbb{T}_0$  is contained in the convex envelope of  $P^1, P^2, P^3$ , then no length can go to zero during the evolution, by Remark 10.16, the maximal time of existence is  $+\infty$  and the triods  $\mathbb{T}_t$  tend, as  $t \rightarrow +\infty$ , to the unique Steiner triod.

In the case that the maximal time  $T$  is finite and a curve collapses to an end–point (see Figure 20 and the above discussion) at the moment we are not able to restart the flow. Indeed, although the curvature is bounded, Theorem 11.1 does not apply and we need some “boundary” extension, see the discussion in Section 10.4 (after Figure 20).

## 15.2 The spoon

In Section 8.2 we discussed the behavior of the area of the bounded regions enclosed by an evolving regular network. In the case of the spoon, the loop is formed by only one curve and there is only one triple junction. Equation (8.2) then gives

$$A'(t) = -\frac{5\pi}{3}.$$

This implies that the maximal time  $T$  of existence of a smooth flow of a spoon is finite and

$$T \leq \frac{3A_0}{5\pi},$$

where  $A_0$  is the initial area enclosed in the loop (see Proposition 13.3).

As  $t \rightarrow T$ , the only possible limit regular shrinkers  $\tilde{\Gamma}_\infty$  arising from Huisken’s rescaling procedure at a reachable point  $x_0 \in \overline{\Omega}$  are given by

- a halfline from the origin,
- a straight line through the origin,
- a standard triod,
- a Brakke spoon (see Figure 6).

This follows by the simple topological structure of  $\Gamma_t$  and the uniqueness (up to rotation) of the Brakke spoon among the shrinkers in its topological class (see Section 8.6). We remind that all the possible blow–up limits are non–degenerate networks with multiplicity one, thank to Corollary 14.11.

We first notice that, if the curve  $\gamma^1$  shrinks, then the curvature clearly cannot be bounded, hence, by Proposition 10.28, it is not possible that both lengths of  $\gamma^1$  and  $\gamma^2$  go to zero, as  $t \rightarrow T$ .

Suppose that the length of the “open” curve  $\gamma^2$  is uniformly positively bounded from below, then the curve  $\gamma^1$  must shrink and the maximum of the curvature goes to  $+\infty$  as  $t \rightarrow T$  (indeed,  $\lim_{t \rightarrow T} \int_{\mathbb{S}_t} k^2 ds = +\infty$ , by Proposition 13.3). Then, if  $x_0 = \lim_{t \rightarrow T} O(t)$ , taking a blow-up  $\tilde{\Gamma}_\infty$  at  $x_0 \in \Omega$  we can only get a Brakke spoon, since in the other cases (a halfline is obviously excluded) the curvature would be locally bounded and the flow regular. Hence, as  $t \rightarrow T$ , the length of the closed curve  $\gamma^1$  goes to zero, the area  $A(t)$  enclosed in the loop goes to zero at  $T = \frac{3A_0}{5\pi}$ , indeed  $A(t) = A_0 - 5\pi t/3$  and  $\Gamma_t$  converges to a limit network composed only by the limit  $C^1$  curve  $\gamma_T^2$  connecting  $P$  with  $x_0$  (and curvature going as  $o(1/d_{x_0})$ ), as in Figure 23.

If instead the length of the curve  $\gamma^2$  is not positively bounded from below for hypothesis, then, as  $t \rightarrow T$ , by Proposition 10.28, such curve collapses to the end-point  $P$ , the curvature stays bounded and the network  $\Gamma_t$  is locally a tree around every point, uniformly in  $t \in [0, T)$ . Hence, the region enclosed by the curve  $\gamma^1$  does not vanishes and the triple junction  $O$  has collapsed onto the boundary point  $P$ , maintaining the 120 degrees condition and bounded curvature (see Proposition 10.21). The networks  $\Gamma_t$  converge in  $C^1$ , as  $t \rightarrow T$ , to a limit network  $\Gamma_T$ , as in Figure 21.

**Theorem 15.3.** *Consider a smooth, embedded, initial spoon  $\Gamma_0$  in a smooth, strictly convex and open set  $\Omega \subset \mathbb{R}^2$ , with a fixed end-point  $P \in \partial\Omega$ , with initial area enclosed in the closed curve equal to  $A_0$ . Then there exists a smooth evolution by curvature  $\Gamma_t$  of  $\Gamma_0$  in a maximal time interval  $[0, T)$  with  $T \leq \frac{3A_0}{5\pi}$ , which at every time is a smooth embedded regular spoon in  $\Omega$ .*

Moreover,

- either the limit of the length of the curve that connects the 3-point the end-point  $P$  goes to zero, as  $t \rightarrow T$ ,  $T < \frac{3A_0}{5\pi}$ , the curvature remains bounded and  $\Gamma_t$  converges to a limit  $C^1$  network, as depicted in Figure 21;
- or the lengths of the curve composing the loop goes to zero, as  $t \rightarrow T$ , in this case  $T = \frac{3A_0}{5\pi}$ , the area  $A(t)$  of the bounded region goes to zero,  $\lim_{t \rightarrow T} \int_{\mathbb{S}_t} k^2 ds = +\infty$ , the limit network  $\Gamma_t$  is composed by a single open  $C^1$  curve  $\gamma_T^2$ , as depicted in Figure 23.

In the second case, at the “free” end-point  $x_0 = \lim_{t \rightarrow T} O(t) \in \Omega$  of the limit curve  $\gamma_T^2$ , for a subsequence of rescaled times  $t_j \rightarrow +\infty$  the associate rescaled networks  $\tilde{\Gamma}_{t_j}$  around  $x_0$  tend in  $C_{\text{loc}}^1 \cap W_{\text{loc}}^{2,2}$  to a Brakke spoon, as  $j \rightarrow \infty$ .

At the moment we do not have a way to restart the flow in the first situation. In the second one, a natural “choice” is to assume that the flow ends and the whole network vanishes for  $t > T$ .

We conclude this example with a couple of open questions.

**Open Problem 15.4** (Special case of Problem 8.22). The limit Brakke spoon obtained in the previous theorem (in the second situation) is independent of the chosen sequence of times  $t_k \rightarrow +\infty$ ? That is, the direction of its unbounded halfline is unique?

**Open Problem 15.5.** Having in mind the “convexification” result for simple closed curves by Grayson [35] (see Remark 2.2), a natural question is: if we consider an initial spoon moving by curvature with the length of the non-closed curve uniformly positively bounded below during the evolution, the closed curve becomes eventually convex and then remains convex?

These two open problems are connected each other, since the uniqueness of the blow-up limit (which is a Brakke spoon, hence with a convex region) would imply that the region at some time becomes convex and then remains so, by the smooth convergence of the rescaled networks to the Brakke spoon (this follows by the argument of Lemma 8.6 in [47], see the discussion just after the proof of Lemma 8.18).

## 16 Open problems

In this section we recall some problems that we find the most important among the several open questions scattered in the text.



1. *Definition of the flow.*

Our “parametric” approach gives a good definition for the curvature flow of a network, compared with the existing notions of generalized evolutions for singular objects, more general but allowing weaker conclusions. The only unsatisfactory point is that we *impose* the presence of only triple junctions and the 120 degrees angle condition. Thanks to them, we have the well-posedness of the system of PDE’s (4.3), hence the short time existence Theorem 4.8.

Nevertheless, one may wonder if these two conditions are automatically satisfied, instantaneously, for every positive time, by choosing a different suitable definition of the curvature flow of a network.

2. *Uniqueness.*

In Theorem 6.8 we showed the existence of solutions to Problem (2.3) for any initial  $C^2$  regular network. A natural problem is the geometric uniqueness (roughly speaking, considering the evolving networks as subsets of  $\mathbb{R}^2$ , see Definition 4.21) of such curvature flow in the class of the curvature flows of networks which are  $C^2$  in space and  $C^1$  in time, see the discussion in the second part of Section 4 and Problem 4.25.

The analogous question can be posed for the curvature flow constructed in Theorem 11.1. Here, the situation is more delicate since for some special initial networks there are surely more than one curvature flow (see Figure 28), but one could hope in a uniqueness result for a (dense) class of “generic” initial networks (Problem 11.29).

3. *Multiplicity–one conjecture.*

Maybe the main open problem in the subject is the multiplicity–one conjecture, that is, whether every blow–up limit shrinker is an embedded network with multiplicity one (see Problem 10.1). Several of the arguments and results in this work depend on such conjecture, we mention its fundamental role in the description of the limit network at a singular time and, consequently, in the possibility to implement the restarting procedure in order to continue the evolution, moreover, it is also a key ingredient in showing that the curvature of a tree–like network is uniformly bounded during the flow for all times and that one has only to deal with “standard transitions” at the singular times (see Section 10).

At the moment, we are able to prove the (strong) multiplicity–one conjecture only for networks with at most two triple junctions (see Section 14).

4. *Uniqueness of blow–up limits.*

According to Proposition 8.20, the sequence of rescaled networks  $\tilde{S}_{x_0, t_j}$  associated to a sequence of rescaled times  $t_j \rightarrow +\infty$ , converges to a degenerate regular shrinker  $\tilde{S}_\infty$ , only up to a subsequence. Analogously, in Proposition 8.16, the sequence of rescaled curvature flows  $S_t^{\mu_i}$  converges to a degenerate regular self–similarly shrinking flow  $S_t^\infty$ , up to a subsequence.

One would like to prove that the limit degenerate regular shrinker  $\tilde{S}_\infty$  (and/or the degenerate regular self–similarly shrinking flow  $S_t^\infty$ ) is actually independent of the chosen converging subsequences, that is, the full family  $\tilde{S}_{x_0, t} C_{loc}^1$ –converges to  $\tilde{S}_\infty$ , as  $t \rightarrow +\infty$ . This is what we called *uniqueness assumption* in Problem 8.22 and it is fundamental for the conclusions of Proposition 10.30 and Theorem 10.37, necessary to restart the flow when a region collapses at a singular time.

5. *Behavior when a region collapses.*

The singularities when a whole region collapses and then vanishes are the most difficult to deal with, in particular because the curvature is unbounded. We are not able, at the moment, to give a complete picture of the behavior of the evolving network, getting close to the singular time. A couple of conjectures are stated in Problems 8.25 and 8.26, in particular, we expect that the non–collapsing curves “exiting” from the collapsing regions (and converging to the concurring curves at the new multi–point of the limit network) have locally uniformly bounded curvature during the flow and that, anyway, such singularities are actually all Type I singularities, see Remark 8.21 (in other words, the curvature flow of embedded networks does not develop Type II singularities).

Anyway, hypothetically admitting the possibility of Type II singularities, one is led to consider and try to analyze/classify also Type II blow–up limits (see [63, Section 7]), which are actually



“eternal” curvature flows of regular networks (for instance, the “translating” ones, see [63, Section 5.2], that possibly coincide with them).

6. *Classification of shrinkers.*

Several questions (also of independent interest) arise in trying to classify the (embedded) regular shrinkers. Such a classification is complete for shrinkers with at most two triple junctions [9, 10], or for the shrinkers with a single bounded region [10, 16, 17, 74], see the following figure.

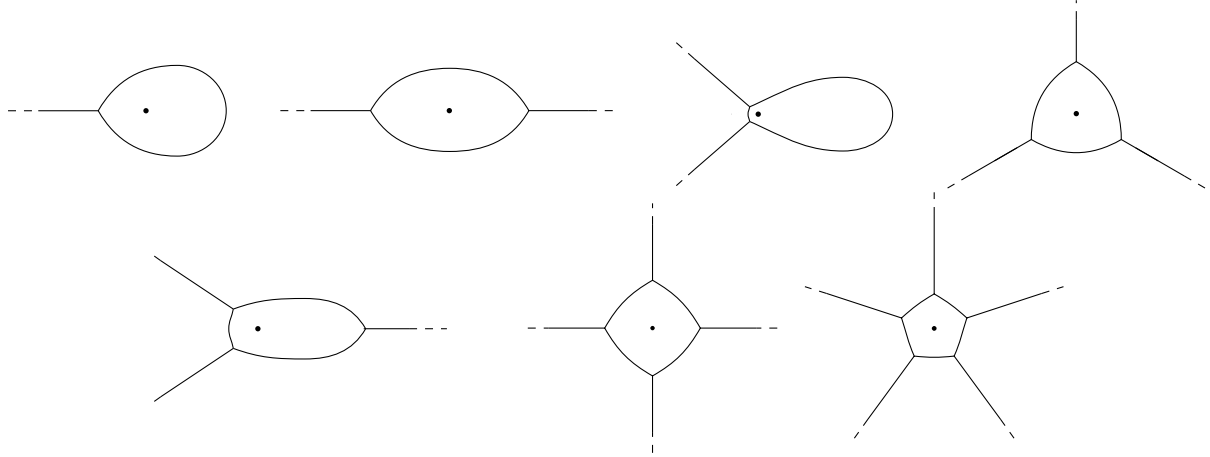


Figure 38: The regular shrinkers with a single bounded region.

A lot of numerical computations, partial results and conjectures can be found in [40].

We mention the very natural question whether there exist regular shrinkers with more than five unbounded halflines.

7. *The set of singular times.*

An important point to be understood in order to define a curvature flow in a maximal time interval, passing through the singular times by means of the restarting procedure (see Section 12), is whether the singular times are discrete, or even finite, or in some situation they can accumulate (Problem 13.1). In this latter case, at the moment we actually do not know how to continue the flow, if it is possible.

8. *Asymptotic convergence.*

In case of global existence in time of a (possibly “extended”, see Section 13) curvature flow, we would like to show the convergence of the evolving network, as  $t \rightarrow +\infty$ , to a stationary network for the length functional (Problem 13.8).

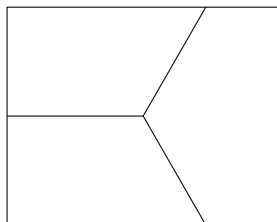
## Appendix – A regular shrinkers gallery (courtesy of Tom Ilmanen)

The following figures of regular shrinkers with their Gaussian density are based on numerical computations due to J. Hättenschweiler (see [40] where one can also find other positive and negative examples and several conjectures) and T. Ilmanen. We remark that this is not an exhaustive list, only the shrinkers with at most one bounded region are completely classified, by the work of Chen and Guo [17] (and actually they are the only ones in this gallery whose existence is rigorously proved). Moreover, all the shrinkers shown below have at least one symmetry axis, we do not know of examples without any symmetries at all.

### No regions:

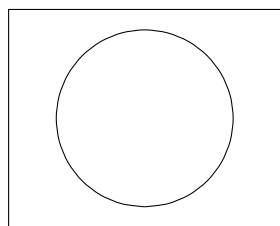


Line  
 $\Theta = 1$

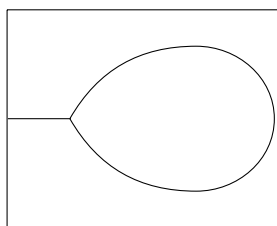


Triod  
 $\Theta = 1.5$

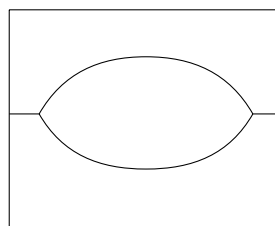
### 1 region:



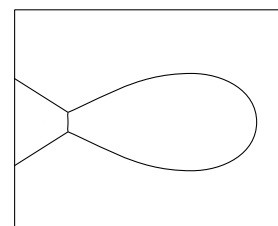
Circle  
 $\Theta = \sqrt{2\pi/e} \approx 1.520$



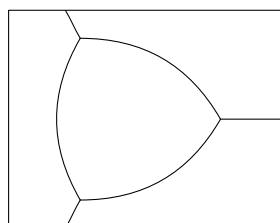
Spoon  
 $\Theta \approx 1.699$



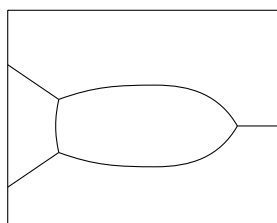
Lens  
 $\Theta \approx 1.789$



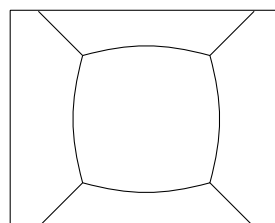
Fish  
 $\Theta \approx 2.026$



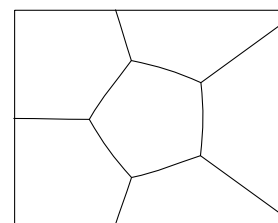
3-ray star  
 $\Theta \approx 2.031$



Rocket  
 $\Theta = ?$

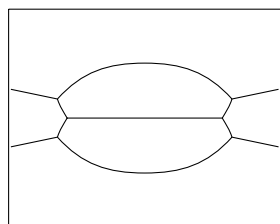


4-ray star  
 $\Theta \approx 2.295$

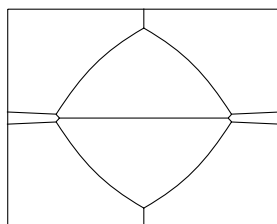


5-ray star  
 $\Theta \approx 2.606$

### 2 regions:

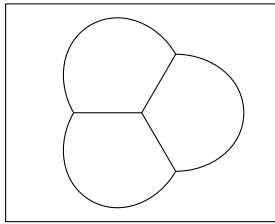


Cisgeminate eye  
 $\Theta = ?$

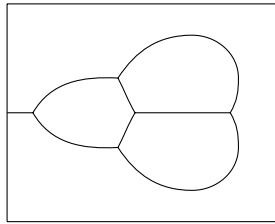


Cisgeminate 4-ray star  
 $\Theta = ?$

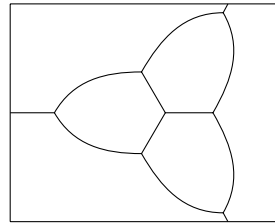
### 3 regions:



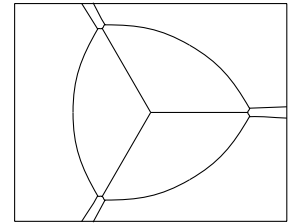
Mercedes-Benz  
 $\Theta \approx 2.532$



1-ray Mercedes-Benz  
 $\Theta \approx 2.598$

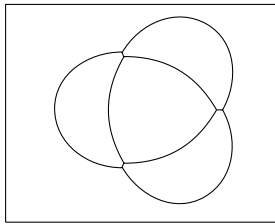


3-ray Mercedes-Benz  
 $\Theta \approx 2.762$

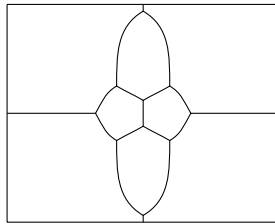


Cisgeminate 3-ray star  
 $\Theta = ?$

### 4 regions:

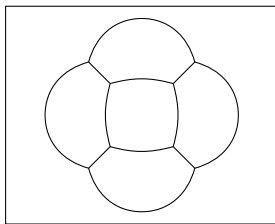


3-leaf clover  
 $\Theta \approx 3.064$

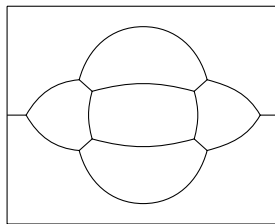


2-ray 2-floc  
 $\Theta \approx 3.249$

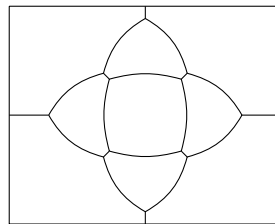
### 5 regions:



4-leaf clover  
 $\Theta \approx 3.234$

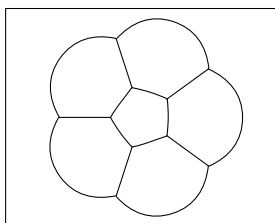


2-ray 4-leaf clover  
 $\Theta \approx 3.365$

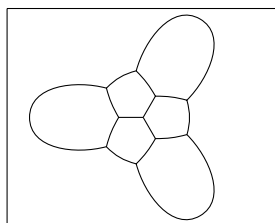


4-petal flower  
 $\Theta \approx 3.474$

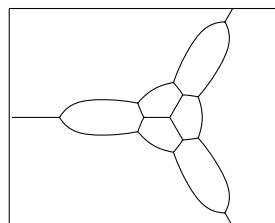
### 6 regions:



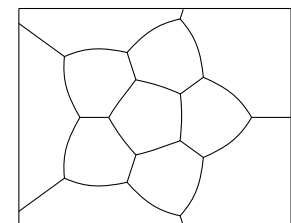
5-leaf clover  
 $\Theta \approx 3.455$



3-floc  
 $\Theta \approx 3.477$

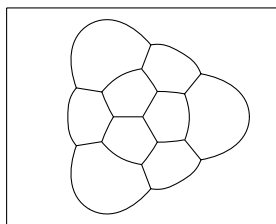


3-ray three-floc  
 $\Theta \approx 3.517$

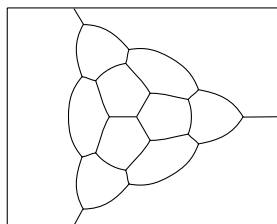


5-petal flower  
 $\Theta \approx 3.907$

9 regions:

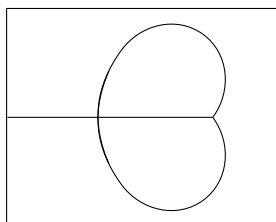


9-floc  
 $\Theta \approx 4.194$

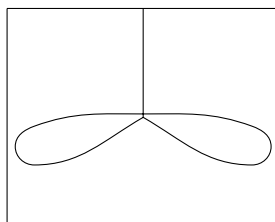


3-ray 9-floc  
 $\Theta \approx 4.321$

Non-embedded regular shrinkers:

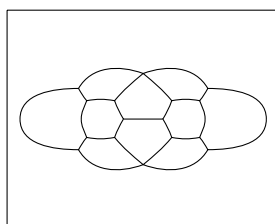
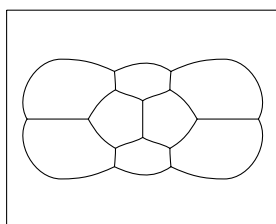
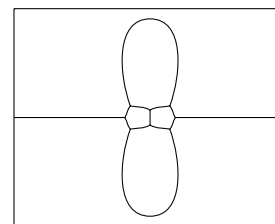
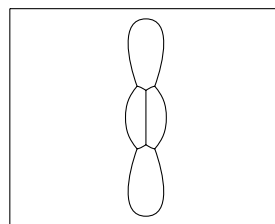
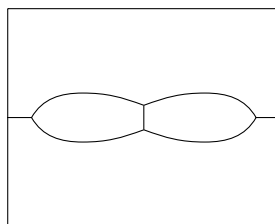
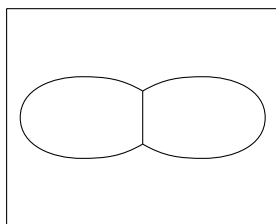


Antispoon  
 $\Theta \approx 2.365$



Bowtie  
 $\Theta \approx 2.503$

Impossible regular shrinkers:



Conjecturally, by numerical evidence in [40], there are no regular shrinkers with these topological shapes. The only one whose non-existence is rigorously proved is the first one, the  $\Theta$ -shaped (double cell) shrinker, in [10].

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